FR-PHENO-2011-XXX

Calculational Explorations in $N = 1$ Superspace

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September 29, 2011

Abstract

This is a write-up of a course on supersymmetry given at the 42nd and 43rd Maria Laach School on High Energy Physics in 2010 and 2011. The course time was shared between little lecture blocks and blocks where students worked on problems, which is reflected in the write-up. The selected material starts with motivating supersymmetry as the unique viable possibility to extend Poincaré symmetry in the context of relativistic quantum field theory. After a reasonable amount of spinor gymnastics, superspace is introduced and everything necessary for the formulation of supersymmetric models with potential relevance at the electroweak scale is constructed from there. Finally, the simplest supersymmetric field theories, namely the Wess Zumino model and supersymmetric Abelian gauge theory, are formulated and discussed.

Preliminary Note

This is a write-up of the course on supersymmetry which I gave at the 42nd and 43rd Maria Laach School on High Energy Physics in Maria Laach Abbey (2010) and Bautzen (2011), respectively. While the course is quite a bit algebra-oriented, it is meant to be accessible for theorists and experimentalists because the required previous knowledge is minimal. Essentially all necessary relations are derived within the course. For this reason, the variety of topics covered in the course is comparably limited. However, it is my hope that a more solid knowledge and understanding of the basics of supersymmetry, as was the aim here, will enable the course participants to further their studies in topics of supersymmetry if they wish.

The course time was shared between little lecture blocks and blocks where students worked on problems, which is reflected in the write-up. In the write-up, problems are in grey-shaded boxes and the answers (and sometimes a few steps of the solution) follow directly afterwards. In total, the designated problems in the write-up are more than one could do in the roughly 12 hours. So a selection was made. In particular, there are two problems which get quite lengthy, when done completely: a) the transformation of a scalar superfield under a supersymmetry transformation and b) the product of two scalar superfields. Both can be shortened considerably, if only the terms most necessary for what comes afterwards are done as problems.

I would like to thank the organisers of the Maria Laach School, Siegfried Betkhe, Thomas Mannel and Reinhold Rückl for inviting and re-inviting me to deliver a course on supersymmetry and for the organisation of such an excellent school. My greatest thanks go to the students which attended my course in 2010 and 2011.

Oliver Brein, Freiburg, 19.9.2011

Contents

1 Motivation

1.1 Coleman Mandula Theorem

In the 1960's, efforts had been made to relate baryons of different spin by symmetries in a relativistically invariant way. This endeavour failed. In fact, it was proven in a sequence of "no-go" theorems that, assuming certain properties of the considered quantum field theory to hold, it does not make sense to try it.

The theorem with the widest applicability is the Coleman Mandula Theorem [1], of which we will only report its main result:

> For a quantum field theory with relativistic invariance, analyticity of scattering matrix elements, and some additional reasonable conditions (see [1]), the broadest symmetry group of the S-matrix must be a *direct product* of the Ponicaré group and a group of internal symmetries. In other words, no non-trivial connection between Ponicaré symmetry and internal symmetry is possible.

The theorem is formulated for Lie algebras, i.e. based on generators of symmetry transformations which are related to each other by *commutation relations*. A Lie algebra with N elements A_i $(i = 1, \dots, N)$ can be written as

$$[A_i, A_j] = i f_{ijk} A_k \,, \tag{1}$$

where the so-called structure constants f_{ijk} are numbers (real or complex) and a sum over k, running from 1 to N, is implied. The elements A_i can be viewed as basis vectors of a vector space. The algebra is precisely this vector space which is spanned by all possible linear combinations of the A_i.

From a given algebra, one can get the mathematical structure of a group by exponentiating elements of the algebra:

$$g(\alpha) = e^{i\alpha_k A_k} \,, \tag{2}$$

with continuous parameters $\alpha \equiv (\alpha_1, \cdots, \alpha_N)$ (sum over k implied again!). Equation (1) expresses that the commutator of any two elements A_i and A_j of the algebra is again an element of the algebra, i.e. a certain linear combination of the "basis vectors" A_i. A group "multiplication" can be given by the Baker-Campbell-Hausdorff (BCH) Formula

$$g(\alpha)g(\beta) \equiv e^A e^B = \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{n!} C_n(A, B) \right\} \tag{3}$$

with $A = i\alpha_k A_k$ and $B = i\beta_k A_k$. It would be tedious, and at this point irrelevant, to try to give an explicit form of the functions $C_n(A, B)$. The essential observation here is that the $C_n(A, B)$ are *sums of* $(n-1)$ *multiple commutators* of A and B. From Eq. (1), it is clear that any multiple commutator of elements of the algebra is also an element of the algebra. Thus, all the $C_n(A, B)$ are elements of the algebra and can be calculated using the commutation relations (1) for the elements A_i.

There is a simple algorithmic way to obtain expressions for the $C_n(A, B)$ explicitly up to a given order. This comprises our first exercise:

Derive C_1, C_2, [C_3] using the Baker-Campbell-Hausdorff Formula in the form

$$\ln(e^A e^B) = \sum_{n=1}^{\infty} \frac{1}{n!} C_n(A,B)\,. \tag{4}$$

Remember that the operator-valued functions $\ln(X)$ and $\exp(X)$ are defined by their Taylor expansion:

$$\ln(X) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{X^n}{n} = X - \frac{X^2}{2} + \frac{X^3}{3} \pm \cdots\,, \tag{5}$$

$$e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!} = 1 + X + \frac{X^2}{2} + \frac{X^3}{3!} + \cdots\,. \tag{6}$$

In order to sort powers of algebra elements more easily, one can use the trick to introduce an arbitrary parameter t by shifting $A \to tA$ and $B \to tB$ in Eq. (4) and use the fact that $C_n(tA, tB) = t^n C_n(A, B)$.

Contrary to the spirit of the exercises, let us reveal the result of the above exercise here (plus a few more terms) for future reference:

$$C_1(A,B) = A + B\,, \tag{7}$$

$$C_2(A,B) = [A,B]\,, \tag{8}$$

$$C_3(A,B) = \frac{1}{2}[A,[A,B]] + \frac{1}{2}[B,[B,A]]\,, \tag{9}$$

$$C_4(A,B) = [[A,[A,B],B]\,. \tag{10}$$

While, by definition, the n-th order term C_n is a polynomial in the (in general) non-commuting elements A and B, it is still remarkable that all those terms combine to a sum of multiple commutators.

1.2 Not covered by Coleman Mandula

As is always the case with "no-go" theorems, their decision on whether some theoretical proposal is a good or a bad idea relies on the assumptions of the theorem. Finding a loophole in the Coleman Mandula Theorem was not how supersymmetry was discovered originally [2, 3, 4]. Yet, it is now the most natural starting point for introducing supersymmetry. So, let's do it this way.

The Coleman Mandula Theorem considers symmetry generators which all fulfil *commutation relations* among each other. Not covered by the theorem are extended algebraic structures, so-called "graded" Lie algebras, which also involve *anti-commutation relations*.

Such structures, contain so-called "bosonic" and "fermionic" elements: B and F, respectively. Schematically, the algebraic relations among those elements have the form:

$$[B, B'] = B'' , \qquad [B, F] = F' , \qquad \{F, F'\} = B , \tag{11}$$

i.e. the commutator of two bosonic elements is again a bosonic element, the commutator of a bosonic and a fermionic element is a fermionic element and the anti-commutator of two fermionic elements is a bosonic element. More explicitly, given some elements B_i and F_α, the algebraic relations can be written as

$$[B_i, B_j] = i a_{ijk} B_k , \qquad [B_i, F_\alpha] = b_{i\alpha\beta} F_\beta , \qquad \{F_\alpha, F_\beta\} = i c_{\alpha\beta i} B_i , \tag{12}$$

with structure constants a_{ijk}, $b_{i\alpha\beta}$ and $c_{\alpha\beta i}$. Again, sums over the doubly appearing indices on the right-hand sides of the three equations are implied.

Not every choice of structure constants gives a valid set of algebraic relations. In fact, the structure constants underlie several constraints which follow from the algebraic structure. Anti-symmetry and symmetry properties of the commutators and anti-commutators under the exchange of elements result in constraints like $a_{ijk} = -a_{jik}$ and $c_{\alpha\beta i} = c_{\beta\alpha i}$. However, there are also identities which hold for certain sums of products of three elements of the graded algebra, the so-called Graded Jacobi Identities:

$$[[B, B'], B''] + [[B'', B], B'] + [[B', B''], B] = 0 , \tag{13}$$
$$[\{F, F'\}, B] + \{[B, F], F'\} - \{[F', B], F\} = 0 , \tag{14}$$
$$[[B, B'], F] + [[B', F], B] + [[F, B], B'] = 0 , \tag{15}$$
$$[\{F, F'\}, F''] + [\{F', F''\}, F] + [\{F'', F\}, F'] = 0 , \tag{16}$$

with B, B', B'' and F, F', F'' any bosonic and fermionic element of the graded algebra, respectively. Writing the Graded Jacobi Identities for explicit elements of the graded algebra B_i and F_α, and using Eqs. (12) repeatedly, one obtains a set of equations which the structure constants are also constrained to fulfil.

In order to demonstrate that the Graded Jacobi Identities just follow from the definition of commutation and anti-commutation brackets, a little exercise is in order: show that Eq. (14) holds [and, if you feel like it, go for Eq. (16) too].

One could worry now whether exponentiation of elements of the graded Lie algebra leads to group elements as in the case of an ordinary Lie algebra. After all, the graded algebra only knows *anti-commutation* relations between the fermionic elements of the algebra, while the BCH Formula (3) always requires the knowledge of *commutation* relations between any element of the algebra. However, the step from the (graded) algebra to group elements by exponentiation, using the Baker-Campbell-Hausdorff Formula (3), can be achieved using *Grassmann numbers* as parameters to multiply the fermionic generators.

Grassmann numbers are algebraic objects which are anti-commuting and nilpotent, i.e. their square vanishes. Given two Grassmann numbers ξ and η, their essential features can be summarised as follows:

$$\xi\eta = -\eta\xi\,, \qquad \xi^2 = \eta^2 = 0\,. \tag{17}$$

Furthermore, in the context of graded Lie algebras we develop here, the Grassmann parameters are required to anti-commute with all fermionic elements of the algebra and commute with all bosonic elements:

$$\xi B = B\xi\,, \qquad \xi F = -F\xi\,. \tag{18}$$

Thus we get for the commutator of fermionic elements F, F' multiplied by Grassmann parameters ξ, η:

$$\begin{aligned}
[\xi F, \eta F'] &= \xi F \eta F' - \eta F' \xi F \\
&= -\xi F F' \eta - \xi \eta F' F \\
&= -\xi F F' \eta - \xi F' F \eta \\
&= -\xi\{F, F'\}\eta = -\xi\eta\{F, F'\}\,,
\end{aligned}$$

i.e. the commutator $[\xi F, \eta F']$ is determined by the anticommutator $\{F, F'\}$. Hence, exponentiation of elements of the graded Lie algebra leads to group elements g, with the group multiplication law governed by the BCH formula, if bosonic elements are multiplied by commuting numbers (say real or complex numbers) and fermionic elements by Grassmann numbers, e.g.

$$g(\alpha, \xi) \equiv g(\alpha_1, \cdots, \xi_1, \cdots) = e^{i(\alpha_k B_k + \xi_\beta F_\beta)}\,.$$

1.3 Haag Łopuszański Sohnius Theorem

After having set up the mathematical stage for this profound evasion of the Coleman Mandula Theorem, one can try to find fermionic elements of a graded Lie algebra, additional to the generators of Poincaré transformations, i.e. translations in time and space, boosts and rotations. Based on the Coleman Mandula Theorem, Haag Łopuszański and Sohnius studied the possibilities for symmetries of the S-matrix of a quantum field theory of such an extended algebraic structure. The fact that the Poincaré generators need to be part of the algebra as well as the constraints coming from the graded Jacobi Identities (13) to (16) restrict possible structures considerably. In fact, the term "supersymmetry algebra" had already been coined before the paper of Haag Łopuszański and Sohnius for spin-$\frac{1}{2}$ operators $Q, \bar{Q}$ fulfilling (schematically) the following algebraic relations with the 4-momentum operator:

$$\{Q, \bar{Q}\} \propto P\,, \qquad [Q, P] = [\bar{Q}, P] = 0\,.$$

With this knowledge, a brief statement of the Haag Łopuszański Sohnius Theorem can now be formulated as follows:

Of all graded Lie algebras, only supersymmetry algebras (together with their extension to include "central charges") give symmetries of the S-matrix which are consistent with relativistic quantum field theory.

The elements of the most general supersymmetry algebra are:

P_μ	generators of translations in time and space ,
$M_{\mu\nu}$	generators of rotations and boosts (i.e. Lorentz transformations) ,
$Q^L_\alpha, \bar{Q}^L_{\dot{\alpha}}$	"super charges," generators of so-called supersymmetry transformations ($L = 1, \cdots, N$) ,
B_l	generators of internal symmetry transformations .

The supersymmetry transformations generated by the Q's and $\bar{Q}$'s should be seen as external symmetries. The consecutive application of two supersymmetry transformations leads to a translation. So, in a sense, the super charges can be regarded as something like square roots of the momentum operator. The open index α and $\dot{\alpha}$ of Q^L and $\bar{Q}^L$, respectively, denote the components of the Weyl spinor representation space Q^L and $\bar{Q}^L$ belong to. This will be explained in the next section. The matrix of generators of Lorentz transformations $M_{\mu\nu}$ is antisymmetric under exchange of the indices ($M_{\mu\nu} = -M_{\nu\mu}$), i.e. there only 6 independent generators: $M_{0i}(i = 1, 2, 3)$ corresponding to boosts in three spatial directions and M_{12}, M_{23}, M_{31} to rotations about three axes. The internal symmetry transformations, generated by the B_l's, form a compact, semi-simple group with possibly extra Abelian factors.

Returning (shortly) to the no-go aspect of the Coleman-Mandula Theorem, i.e. the forbidden mixing between external and internal symmetries, one can state now in what respect such a mixing can happen in the framework of supersymmetry. If there are two or more super charges (i.e. $N > 1$), they may transform non-trivially under internal symmetry transformations according to some representation $r(B_l)$:

$$[Q^L_\alpha, B_l] = i\,[r(B_l)]^L{}_M Q^M_\alpha\,, \qquad\qquad [\bar{Q}^L_{\dot{\alpha}}, B_l] = i\,[r(B_l)^\star]^L{}_M \bar{Q}^M_{\dot{\alpha}}\,.$$

We refrain here from showing the full set of algebraic relations of the general supersymmetry algebra for arbitrary N as we will focus in the rest of the course on the case $N = 1$.

There are many reasons to bring forward which show that ($N > 1$) supersymmetry does not allow to construct models which embed our current understanding of particle physics (the Standard Model, that is). Trying to implement models with supersymmetric features, which are at the same time valid at the electroweak scale, the observation of electroweak symmetry is mandatory. The smallest building blocks of ($N > 1$) supersymmetry, however, are field multiplets (so-called super multiplets) which contain left- *and* right-chiral fermion fields among their components. Those component fields are mapped into linear combinations of them under supersymmetry transformations. Hence, the chirality of the electroweak interactions, which requires the left- and right-chiral fermions to transform according to different representations of the SU(2)×U(1) gauge group, cannot be consistent with supersymmetry in this case. Only for $N = 1$ supersymmetry, there exist super multiplets which contain just a left-chiral or just a right-chiral fermion field among their fermionic components. As our focus here is on the construction of models with supersymmetric features which are phenomenologically relevant down to the electroweak scale, we will discuss $N = 1$ supersymmetry in the following. However, the argument for $N = 1$ as the only viable possibility for supersymmetry at the electroweak scale does not exclude the possibility of having ($N > 1$) supersymmetry realized in nature at some (possibly much) higher energy scale.

Let us present now the main result of the Haag Łopuszański Sohnius paper for the $N = 1$ case: the $N = 1$ Super-Poincaré Algebra: It, of course, contains the relations of the Poincaré Algebra:

$$[M_{\mu\nu}, M_{\rho\lambda}] = i(g_{\mu\rho}M_{\lambda\nu} - g_{\mu\lambda}M_{\rho\nu} + g_{\nu\rho}M_{\mu\lambda} - g_{\nu\lambda}M_{\mu\rho}) \,, \tag{19}$$

$$[P_\mu, M_{\nu\rho}] = i(g_{\mu\nu}P_\rho - g_{\mu\rho}P_\nu) \,, \tag{20}$$

$$[P_\mu, P_\nu] = 0 \,. \tag{21}$$

Furthermore, there are the relations of the super charges among themselves and with the Poincaré generators:

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\,\sigma^\mu_{\alpha\dot{\alpha}}P_\mu \,, \tag{22}$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \,, \tag{23}$$

$$[Q_\alpha, P_\mu] = [\bar{Q}_{\dot{\alpha}}, P_\mu] = 0 \,, \tag{24}$$

$$[Q_\alpha, M_{\mu\nu}] = \frac{1}{2}(\sigma_{\mu\nu})_\alpha^{\ \beta} Q_\beta \,, \tag{25}$$

$$[\bar{Q}_{\dot{\alpha}}, M_{\mu\nu}] = -\frac{1}{2}\bar{Q}_{\dot{\beta}}(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}_{\ \dot{\alpha}} \,. \tag{26}$$

The generators of the internal symmetry group (e.g. the gauge group of the model) fulfil Lie algebra relations

$$[B_l, B_m] = i\, c^k_{lm} B_k \,, \tag{27}$$

and there is also the possibility that the supercharges are charged under some Abelian subgroup of the internal symmetries, i.e. for a certain linear combination $R := a^l B_l$, adjusting the overall normalisation accordingly, one can have an additional so-called R-symmetry:

$$[Q_\alpha, R] = Q_\alpha \,, \qquad [\bar{Q}_{\dot{\alpha}}, R] = -\bar{Q}_{\dot{\alpha}} \,. \tag{28}$$

2 Conventions & Basic Algebraic Relations

2.1 Minkowski Space

The set of all transformations of space and time coordinates which leave the "line element" of the four-dimensional space-time,

$$ds^2 = dt^2 - d\vec{x}^2 = dx_\mu dx^\mu = g_{\mu\nu} dx^\mu dx^\nu \,, \tag{29}$$

invariant are called Poincaré transformations. The Poincaré transformations comprise:

- 4 translations (3 in spatial direction, one in the time direction)
  $x^\mu \to x'^\mu = x^\mu + a^\mu$,
- 6 proper Lorentz transformations (3 rotations, 3 boosts):
  $x^\mu \to x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu$ (with $\det \Lambda = 1$),
- space reflections ($\vec{x} \to -\vec{x}$), time reversal ($t \to -t$).

Any Ponicaré transformation can be represented by consecutive applications of the above transformations. Mathematically speaking, all Ponicaré transformations taken together form a group: the Ponicaré group. The proper Lorentz transformations in combination with space reflections and time reversal form the Lorentz group, a subgroup of the Ponicaré group. The translations together with the proper Lorentz transformations form a Lie group with 10 real continuous parameters.
Let us collect in the following our notation and some essential definitions

– Four-vector: $x = (x^\mu) = (x^0, x^1, x^2, x^3) = (t, \vec{x})$: (contravariant components x).

– Metric tensor $g = (g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$,

$g^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ $=: (g^{\mu\nu})$. With that, we have $(g^{-1}g)^\mu{}_\nu = \sum_{\rho=0}^{3} g^{\mu\rho} g_{\rho\nu} = \delta^\mu{}_\nu$.

– Index naming convention:
 ⋆ Latin letters are space indices $(i, j, k, l, m, \cdots) \in \{1, 2, 3\}$
 ⋆ letters from the back part of the Greek alphabet are
  Lorentz indices $(\mu, \nu, \rho, \sigma, \cdots) \in \{0, 1, 2, 3\}$
 ⋆ letters from the beginning of the Greek alphabet are
  spinor indices $(\alpha, \beta, \gamma, \delta, \cdots) \in \{1, 2\}$ (to be discussed later)

– (Einstein's) summation convention: if the same index appears twice in an expression,

a sum over this index over its appropriate index range is implied,
i.e. $x_\mu y^\mu = \sum_{\mu=0}^{3} x_\mu y^\mu$, $k^l x^l = \sum_{\mu=1}^{3} k^l x^l$, or $\psi^\alpha \chi_\alpha = \sum_{\alpha=1}^{2} \psi^\alpha \chi_\alpha$.

– Definition of covariant indices (by "contraction with the metric"):
$x_\mu := g_{\mu\nu} x^\nu$, $(x_\mu) = (x^0, -x^1, -x^2, -x^3) = (t, -\vec{x})$. Note that $\vec{x}$ always means (x^1, x^2, x^3).

– The change from co- to contravariant indices for any Lorentz tensor is defined by the contraction of the appropriate index with the metric tensor, e.g.
$T^\mu{}_\nu = g^{\mu\rho} T_{\rho\nu}$ oder $T_\mu{}^\nu = g_{\mu\rho} g^{\nu\sigma} T^\rho{}_\sigma$. In particular, $g^\mu{}_\nu = g^{\mu\rho} g_{\rho\nu} = \delta^\mu{}_\nu$.

– Derivative $\partial_\mu := \frac{\partial}{\partial x^\mu}$ and $\partial^\mu := \frac{\partial}{\partial x_\mu} = g^{\mu\nu} \partial_\nu$.
Note the consistency of the index positions: $\partial_\mu x^\mu = \partial^\mu x_\mu = 4$ (= invariant).

– The scalar product

$$x \cdot y = x_\mu y^\mu = g_{\mu\nu} x^\mu y^\nu \,, \tag{30}$$

is invariant under Lorentz transformations. From this, one can read off the major property of Lorentz transformations: $x \cdot y \stackrel{\Lambda}{\to} x' \cdot y' = g_{\mu\nu} x'^\mu y'^\nu = g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma x^\rho y^\sigma$ for any four-vectors x and y. Hence,

$$g_{\rho\sigma} = g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \,. \tag{31}$$

Multiplication of this equation with the inverse matrix $(\Lambda^{-1})^\sigma{}_\tau$ results in:

$$(\Lambda^{-1})^\sigma{}_\tau = g_{\tau\mu} g^{\sigma\rho} \Lambda^\mu{}_\rho = \Lambda_\tau{}^\sigma \,. \tag{32}$$

From this follows

$$\det \Lambda^{-1} = \det \Lambda \qquad \Rightarrow \qquad (\det \Lambda)^2 = 1 \qquad \Rightarrow \qquad \det \Lambda \in \{+1, -1\} \,. \tag{33}$$

Lorentz transformations with $\det \Lambda = -1$ can be represented by a successive application of a proper Lorentz transformation and a space reflection or a time reversal.

2.2 Spinor Space

Spinor Representations of the Proper Lorentz Group

The lowest dimensional, faithful representations of the proper Lorentz group are two-dimensional so-called spinor representations: (2×2)-matrices $M \in \mathrm{SL}(2, \mathbf{C})$ acting on so-called Weyl spinors. The group $\mathrm{SL}(2, \mathbf{C})$ is formed by the set of all (2×2)-matrices M with complex valued entries which fulfil $\det M = 1$. Weyl spinors are 2-component vectors with complex components. In general, a (matrix) representation of a group G is a map

$$\left\{ \begin{array}{lll} G & \longrightarrow & \text{vector space of } (n \times n) \text{ matrices } M \\ g & \mapsto & M(g) \end{array} \right. , \tag{34}$$

where the group "multiplication" ($\circ$) is mapped on the multiplication of $(n \times n)$ matrices, i.e.

$$M(g_2 \circ g_1) = M(g_2)\, M(g_1)\,, \tag{35}$$
$$M(e) = 1\,, \tag{36}$$

where e means the neutral element of the group (i.e. $e \circ g = g \circ e = g$ for any $g \in G$) and 1 the identity matrix. It is easy to see, that if $M(g)$ is a matrix representation of a group, so is $M^\star(g)$, $M^{-1,T}(g)$ and $M^{-1,\dagger}(g)$.

Let us collect now the notation for those four standard spinor representations of the proper Lorentz group. Apart from how the representations are related to each other, the following equations shows our conventions for the positions and naming of spinor indices of the representation matrices and of a corresponding sample Weyl spinor.

$$(\frac{1}{2},0) : M_\alpha{}^\beta\,,\ \psi_\alpha \qquad \leftarrow \text{complex conjugation} \rightarrow \quad (M^\star)_{\dot\alpha}{}^{\dot\beta}\,,\ \bar\psi_{\dot\alpha} \tag{37}$$

$$\Updownarrow \qquad\qquad\qquad\qquad\qquad\qquad \Updownarrow$$

$$(M^{-1,T})^\gamma{}_\delta\,,\ \psi^\alpha \qquad \leftarrow \text{complex conjugation} \rightarrow \quad (M^{-1,\dagger})^{\dot\gamma}{}_{\dot\delta}\,,\ \bar\psi^{\dot\alpha}\ : (0,\frac{1}{2}) \tag{38}$$

While the left and right hand sides in (37) and (38) are inequivalent representations, related to each other by complex conjugation, the lower and upper representations on both sides are equivalent. They can be transformed into each other by similarity transformations with regular matrices ϵ and $\bar\epsilon$ (to be defined below):

$$(M^{-1,T})^\gamma{}_\delta = \epsilon^{\gamma\alpha} M_\alpha{}^\beta (\epsilon^{-1})_{\beta\delta}\,, \tag{39}$$
$$(M^{-1,\dagger})^{\dot\gamma}{}_{\dot\delta} = \bar\epsilon_{\dot\delta\dot\beta} (M^\star)_{\dot\alpha}{}^{\dot\beta} (\bar\epsilon^{-1})^{\dot\alpha\dot\gamma}\,, \tag{40}$$

where a summation over any doubly appearing spinor index is understood.

The conversion of a Weyl spinor with upper or lower indices to one with the opposite index position is also achieved by applying the matrices of the similarity transformations above: the spinor space epsilon tensors (to be defined shortly):

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta\,, \qquad \psi_\alpha = \psi^\beta \epsilon_{\beta\alpha} \qquad \searrow \text{"North-West" rule} \tag{41}$$
$$\psi^{\dot\alpha} = \psi_{\dot\beta}\bar\epsilon^{\dot\beta\dot\alpha}\,, \qquad \psi_{\dot\alpha} = \bar\epsilon_{\dot\alpha\dot\beta}\psi^{\dot\beta} \qquad \nearrow \text{"South-West" rule}\,. \tag{42}$$

The next thing to define is how we will denote complex conjugated objects :

$$(\psi^\alpha)^\star = \bar\psi^{\dot\alpha}\,, \qquad (\psi_\alpha)^\star = \bar\psi_{\dot\alpha}\,. \tag{43}$$

Here, and throughout the course, we only deal with Grassmann-valued Weyl spinors. For such objects, one defines a reversal of the order of factors in products as part of the complex conjugation operation:

$$(\psi^\alpha \chi_\beta)^\star = \bar\chi_{\dot\beta}\bar\psi^{\dot\alpha} \tag{44}$$

and likewise for any other index position.

The spinor-space epsilon tensors are defined as follows:

$$\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}\,, \qquad \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}\,, \qquad \bar{\epsilon}^{\dot{\alpha}\dot{\beta}} = -\bar{\epsilon}^{\dot{\beta}\dot{\alpha}}\,, \qquad \bar{\epsilon}_{\dot{\alpha}\dot{\beta}} = -\bar{\epsilon}_{\dot{\beta}\dot{\alpha}}\,, \tag{45}$$

i.e. they are antisymmetric under exchange of the indices, and the components are

$$\epsilon_{12} = \epsilon^{12} = 1 \qquad \bar{\epsilon}_{\dot{1}\dot{2}} = \bar{\epsilon}^{\dot{1}\dot{2}} = -1\,. \tag{46}$$

Written as a 2×2-matrix, the epsilon tensors look like that:

$$((\epsilon_{\alpha\beta})) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = ((\epsilon^{\alpha\beta})) = -((\bar{\epsilon}_{\dot{\alpha}\dot{\beta}})) = -((\bar{\epsilon}^{\dot{\alpha}\dot{\beta}}))\,. \tag{47}$$

Applying this matrix on a real 2-dimensional vector v, yields a vector $\tilde{v}$ orthogonal to v. Hence, it is easy to understand that $\tilde{v}$ has the same transformation properties with respect to Lorentz transformations as v.

The following properties hold for the spinor-space epsilon tensors:

The ϵ tensors are numerically invariant under Lorentz transformations M, e.g.

$$\epsilon_{\alpha\beta} \xrightarrow{M} M_\alpha{}^\gamma M_\beta{}^\delta \epsilon_{\gamma\delta} = \det M\ \epsilon_{\alpha\beta} = \epsilon_{\alpha\beta}\,. \tag{48}$$

Prove $M_\alpha{}^\gamma M_\beta{}^\delta \epsilon_{\gamma\delta} = \det M\ \epsilon_{\alpha\beta}$.

The upper and lower index versions of the epsilon tensors represent the inverse matrix of each other up to a minus sign:

$$\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = -\delta^\alpha_\gamma\,, \qquad \bar{\epsilon}^{\dot{\alpha}\dot{\beta}}\bar{\epsilon}_{\dot{\beta}\dot{\gamma}} = -\delta^{\dot{\alpha}}_{\dot{\gamma}}\,. \tag{49}$$

Thus, the conversion to epsilon tensors with opposite index position is consistent with the one for spinors:

$$\epsilon^{\alpha\beta}\epsilon^{\delta\gamma}\epsilon_{\beta\gamma} = \epsilon^{\alpha\delta}\,, \qquad \bar{\epsilon}_{\dot{\beta}\dot{\gamma}}\bar{\epsilon}^{\dot{\beta}\dot{\alpha}}\bar{\epsilon}^{\dot{\gamma}\dot{\delta}} = \bar{\epsilon}^{\dot{\alpha}\dot{\delta}}\,. \tag{50}$$

As $((\epsilon_{\alpha\beta}))$ and $((\bar{\epsilon}_{\dot{\alpha}\dot{\beta}}))$ are defined to be off by a factor -1 and both matrices have only real entries, a peculiar-looking rule applies for complex conjugation:

$$(\epsilon_{\alpha\beta})^\star = -\bar{\epsilon}_{\dot{\alpha}\dot{\beta}}\,. \tag{51}$$

The transformation properties of Weyl spinors under a Lorentz transformation Λ are determined by the matrix $M \equiv M(\Lambda)$ as follows:

$$\begin{aligned}
&\psi_\alpha \to \psi'_\alpha = M_\alpha{}^\beta \psi_\beta\,,\\
&\psi^\alpha \to \psi'^\alpha = (M^{-1,T})^\alpha{}_\beta \psi^\beta = \psi^\beta (M^{-1})_\beta{}^\alpha\,,\\
&\bar{\psi}_{\dot{\alpha}} \to \bar{\psi}'_{\dot{\alpha}} = (M^\star)_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\psi}_{\dot{\beta}} = \bar{\psi}_{\dot{\beta}} (M^\dagger)^{\dot{\beta}}{}_{\dot{\alpha}}\,,\\
&\bar{\psi}^{\dot{\alpha}} \to \bar{\psi}'^{\dot{\alpha}} = (M^{-1,\dagger})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\psi}^{\dot{\beta}}\,.
\end{aligned} \tag{52}$$

Show that the spinor products $\phi^\alpha \psi_\alpha$ and $\bar{\phi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}$ are invariant under Lorentz transformations. (Summation over doubly appearing spinor indices is understood.)

Spinor Relations

Products of Weyl spinors where the same number of upper and lower indices appear and which are all contracted (i.e. summed over) are Lorentz invariant. In order to get rid of contracted indices in the notation, we define a standard order of objects with contracted indices:

$$\psi\chi := \psi^\alpha \chi_\alpha \qquad \searrow \text{“North-West” rule}\,, \tag{53}$$

$$\bar{\psi}\bar{\chi} := \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} \qquad \nearrow \text{“South-West” rule}\,, \tag{54}$$

i.e. unwritten contractions of undotted spinor indices are meant to be done in “north-westerly” fashion and of dotted spinor indices in “south-westerly” fashion. This is akin to the rules introduced in the context of “pulling indices” up or down in Eqs. (41) and (42).

A little exercise to get practice with this notation: Show for Weyl spinors $(\psi, \chi, \bar{\psi}, \bar{\chi})$ with anti-commuting (i.e. Grassmann) components that

$$\psi\chi = \chi\psi \qquad \text{and} \qquad \bar{\psi}\bar{\chi} = \bar{\chi}\bar{\psi}. \tag{55}$$

The first relation is proven as follows:

$$\psi\chi = \psi^\alpha \chi_\alpha \overset{\text{Grassmann}}{=} -\chi_\alpha \psi^\alpha \overset{(41)}{=} -\chi^\beta \epsilon_{\beta\alpha} \psi^\alpha \overset{(45)}{=} \chi^\beta \psi^\alpha \epsilon_{\alpha\beta} \overset{(41)}{=} \chi^\beta \psi_\beta = \chi\psi\,. \;\square$$

Now from the nice notation back to ugly-looking explicit components: Given are a Grassmann-valued Weyl spinor $((\theta^\alpha)) = (\theta^1, \theta^2)$ and its complex conjugate $((\bar{\theta}^\alpha)) = (\bar{\theta}^{\dot{1}}, \bar{\theta}^{\dot{2}})$. Calculate $\theta\theta$ $[\bar{\theta}\bar{\theta}]$ explicitly in terms of the components θ^1, θ^2 $[\bar{\theta}^{\dot{1}}, \bar{\theta}^{\dot{2}}]$.

Let us just quote the results here:

$$\theta\theta = -2\theta^1\theta^2 = -2\theta_1\theta_2\,, \qquad \bar{\theta}\bar{\theta} = 2\bar{\theta}^{\dot{1}}\bar{\theta}^{\dot{2}} = 2\bar{\theta}_{\dot{1}}\bar{\theta}_{\dot{2}}\,. \tag{56}$$

These expressions can be used together with the usual anti-commutation properties of Grassmann numbers, $\theta_1\theta_2 = -\theta_2\theta_1$, $(\theta_1)^2 = (\theta_2)^2 = 0$, to solve the next exercise.

Show

$$\theta_\alpha \theta_\beta = -\frac{1}{2}\epsilon_{\alpha\beta}(\theta\theta)\,. \qquad (57)$$

Derive similar expressions for $\theta^\alpha\theta^\beta$, $\bar\theta_{\dot\alpha}\bar\theta_{\dot\beta}$ and $\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta}$.

A write-up is a write-up (sigh). This is what one should arrive at in the other three cases:

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}(\theta\theta)\,, \qquad (58)$$

$$\bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = -\frac{1}{2}\bar\epsilon_{\dot\alpha\dot\beta}(\bar\theta\bar\theta)\,, \qquad (59)$$

$$\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = -\frac{1}{2}\bar\epsilon^{\dot\alpha\dot\beta}(\bar\theta\bar\theta)\,. \qquad (60)$$

The fact that all the above four equations have the same sign is one of the nice features of our choice of definitions for the spinor space epsilon tensors.

Grassmann-valued derivatives (integration)

Derivatives with respect to Grassmann numbers have to be Grassmann objects themselves for consistency reasons. A certain freedom in how to define these objects exists, which is (unfortunately) also reflected in the literature. However, care has to be taken because a consistent definition will not have all properties one would expect to hold from experience with commuting objects.

First let us define the shorthands:

$$\partial_\alpha := \frac{\partial}{\partial\theta^\alpha}\,, \qquad \partial^\alpha := \frac{\partial}{\partial\theta_\alpha}\,, \qquad \bar\partial_{\dot\alpha} := \frac{\partial}{\partial\bar\theta^{\dot\alpha}}\,, \qquad \bar\partial^{\dot\alpha} := \frac{\partial}{\partial\bar\theta_{\dot\alpha}}\,. \qquad (61)$$

We will require (and this is already a debatable choice!) the right-action of those Grassmann derivatives on objects linear in θ or $\bar\theta$ to act as one would expect from commuting variables:

$$\partial_\alpha\theta^\beta = \delta_\alpha^\beta\,, \qquad \partial^\alpha\theta_\beta = \delta_\beta^\alpha\,, \qquad \bar\partial_{\dot\alpha}\bar\theta^{\dot\beta} = \delta_{\dot\alpha}^{\dot\beta}\,, \qquad \bar\partial^{\dot\alpha}\bar\theta_{\dot\beta} = \delta_{\dot\beta}^{\dot\alpha}\,. \qquad (62)$$

Using Eq. (62), show (without changing index positions of the derivatives!) that:

$$\partial_\alpha\theta_\beta = \epsilon_{\alpha\beta}\,, \qquad \partial^\alpha\theta^\beta = -\epsilon^{\alpha\beta}\,, \qquad \bar\partial_{\dot\alpha}\bar\theta_{\dot\beta} = -\bar\epsilon_{\dot\alpha\dot\beta}\,, \qquad \bar\partial^{\dot\alpha}\bar\theta^{\dot\beta} = \bar\epsilon^{\dot\alpha\dot\beta}\,. \qquad (63)$$

Grassmann-valued derivatives have to be anti-commuting too:

$$\partial_\alpha \partial_\beta = -\partial_\beta \partial_\alpha \,, \qquad \bar{\partial}_{\dot\alpha} \bar{\partial}_{\dot\beta} = -\bar{\partial}_{\dot\beta} \bar{\partial}_{\dot\alpha} \,, \qquad \partial_\alpha \bar{\partial}_{\dot\beta} = -\bar{\partial}_{\dot\beta} \partial_\alpha \,, \qquad \text{etc.} \tag{64}$$

Now to one of the unusual properties. The definitions Eqs. (61) and (62) imply

$$\partial^\alpha = -\epsilon^{\alpha\beta} \partial_\beta \,, \qquad \bar{\partial}^{\dot\alpha} = -\bar{\partial}_{\dot\beta} \bar{\epsilon}^{\dot\beta\dot\alpha} \,. \tag{65}$$

We prove here one of the relations:

$$\begin{aligned} \text{Eq. (62):} \quad \partial_\alpha \theta^\beta &= \delta_\alpha^\beta \\ \epsilon^{\gamma\alpha} \partial_\alpha \theta^\beta \epsilon_{\beta\delta} &= \epsilon^{\gamma\alpha} \epsilon_{\alpha\delta} \\ (\epsilon^{\gamma\alpha} \partial_\alpha) \theta_\delta &= -\delta_\delta^\gamma \,. \end{aligned}$$

Comparing this result with $\partial^\gamma \theta_\delta = \delta_\delta^\gamma$ from Eq. (62), we get indeed

$$\partial^\gamma = -\epsilon^{\gamma\alpha} \partial_\alpha \,. \; \square$$

Thus, special care has to be taken when in calculations the index positions of the Grassmann derivatives are changed.

Also seemingly unusual is the behaviour of Grassmann derivatives under complex conjugation:

$$\begin{aligned} \left(\frac{\partial}{\partial \theta^\alpha}\right)^\star &= -\frac{\partial}{\partial \bar\theta^{\dot\alpha}} \,, & \left(\frac{\partial}{\partial \bar\theta^{\dot\alpha}}\right)^\star &= -\frac{\partial}{\partial \theta^\alpha} \,, \\ \left(\frac{\partial}{\partial \theta_\alpha}\right)^\star &= -\frac{\partial}{\partial \bar\theta_{\dot\alpha}} \,, & \left(\frac{\partial}{\partial \bar\theta_{\dot\alpha}}\right)^\star &= -\frac{\partial}{\partial \theta_\alpha} \,, \end{aligned} \tag{66}$$

where all derivatives are meant to act to the right.

Prove Eqs. (66). They follow directly out of the reversal of the order of Grassmann factors under complex conjugation in Eq. (43), which also has to flip the direction in which a Grassmann derivative acts.
For instance, for a Grassmann number ξ, one has $1 = (\frac{\overrightarrow{\partial}}{\partial\xi}\xi) = \xi^\star(\frac{\overleftarrow{\partial}}{\partial\xi})^\star$.

Using Eq. (62) we get the relations:

$$\partial_\alpha \theta^\alpha = \partial^\alpha \theta_\alpha = \bar{\partial}_{\dot\alpha} \bar\theta^{\dot\alpha} = \bar{\partial}^{\dot\alpha} \bar\theta_{\dot\alpha} = 2 \,. \tag{67}$$

Furthermore, a "Grassmannised" Leibniz rule applies:

$$\partial_\alpha (\theta^\beta \theta^\gamma) = \delta_\alpha^\beta \theta^\gamma - \theta^\beta \delta_\alpha^\gamma \,, \tag{68}$$

$$\bar{\partial}_{\dot\alpha} (\bar\theta^{\dot\beta} \bar\theta^{\dot\gamma}) = \delta_{\dot\alpha}^{\dot\beta} \bar\theta^{\dot\gamma} - \bar\theta^{\dot\beta} \delta_{\dot\alpha}^{\dot\gamma} \,. \tag{69}$$

Using the above, show:

$$\partial_\alpha(\theta\theta) = +2\theta_\alpha\,, \qquad \partial^\alpha(\theta\theta) = -2\theta^\alpha\,, \qquad \partial^\alpha\partial_\alpha(\theta\theta) = 4\,, \tag{70}$$
$$\bar\partial_{\dot\alpha}(\bar\theta\bar\theta) = -2\bar\theta_{\dot\alpha}\,, \qquad \bar\partial^{\dot\alpha}(\bar\theta\bar\theta) = +2\bar\theta^{\dot\alpha}\,, \qquad \bar\partial^{\dot\alpha}\bar\partial_{\dot\alpha}(\bar\theta\bar\theta) = 4\,. \tag{71}$$

Generators of SL(2,C)

Proper Lorentz transformations in spinor space are represented by the following matrices

$$(\frac{1}{2},0):\qquad M(\omega)_\alpha{}^\beta = e^{\frac{i}{2}\omega_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\beta}\ \text{ with }\ (\sigma^{\mu\nu})_\alpha{}^\beta = \frac{i}{2}[(\sigma^\mu)_{\alpha\dot\beta}(\bar\sigma^\nu)^{\dot\beta\beta} - (\sigma^\nu)_{\alpha\dot\beta}(\bar\sigma^\mu)^{\dot\beta\beta}]\,, \tag{72}$$

$$(0,\frac{1}{2}):\ (M^{-1,\dagger}(\omega))^{\dot\alpha}{}_{\dot\beta} = e^{\frac{i}{2}\omega_{\mu\nu}(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}}\ \text{ with }\ (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = \frac{i}{2}[(\bar\sigma^\mu)^{\dot\alpha\beta}(\sigma^\nu)_{\beta\dot\beta} - (\bar\sigma^\nu)^{\dot\alpha\beta}(\sigma^\mu)_{\beta\dot\beta}]\,, \tag{73}$$

where

$$((\sigma^\mu)_{\alpha\dot\beta}) = (\mathbf{1}_{\alpha\dot\beta}, (\sigma^k)_{\alpha\dot\beta})\,, \tag{74}$$
$$((\bar\sigma^\mu)^{\dot\alpha\beta}) = (\mathbf{1}^{\dot\alpha\beta}, -(\sigma^k)^{\dot\alpha\beta})\,, \tag{75}$$

and $\mathbf{1}$ is the unit matrix and σ^k are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{76}$$

The Pauli matrices (and of course the unit matrix) are unitary. Hence, under hermitian conjugation, we have

$$(\sigma^\mu)^\dagger = \sigma^\mu \tag{77}$$

or explicitly for the components

$$((\sigma^\mu)_{\alpha\dot\beta})^\star = (\sigma^\mu)_{\beta\dot\alpha}\,. \tag{78}$$

The “bared” and “unbared” Pauli matrices are related to each other via the spinor-space epsilon tensors:

$$(\bar\sigma^\mu)^{\dot\alpha\beta} = \epsilon^{\beta\gamma}(\sigma^\mu)_{\gamma\dot\delta}\bar\epsilon^{\dot\delta\dot\alpha}\,, \tag{79}$$
$$(\sigma^\mu)_{\alpha\dot\beta} = \bar\epsilon_{\dot\beta\dot\gamma}(\bar\sigma^\mu)^{\dot\gamma\delta}\epsilon_{\delta\alpha}\,, \tag{80}$$

and matrix products of them fulfil

$$(\sigma^\mu)_{\alpha\dot\beta}(\bar\sigma^\nu)^{\dot\beta\beta} = g^{\mu\nu}\delta_\alpha^\beta - i(\sigma^{\mu\nu})_\alpha{}^\beta\,, \tag{81}$$
$$(\bar\sigma^\mu)^{\dot\alpha\beta}(\sigma^\nu)_{\beta\dot\beta} = g^{\mu\nu}\delta_{\dot\beta}^{\dot\alpha} - i(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\,. \tag{82}$$

Show that $(\sigma^{\mu\nu})_\alpha{}^\alpha = (\bar{\sigma}^{\mu\nu})^{\dot{\beta}}{}_{\dot{\beta}} = 0$. Use this result, together with the above relations, to calculate the traces

$$(\sigma^\mu)_{\alpha\dot{\beta}}(\bar{\sigma}^\nu)^{\dot{\beta}\alpha} \qquad \text{and} \qquad (\bar{\sigma}^\mu)^{\dot{\alpha}\beta}(\sigma^\nu)_{\beta\dot{\alpha}} \,.$$

For future reference, here's the result of the above exercise:

$$(\sigma^\mu)_{\alpha\dot{\beta}}(\bar{\sigma}^\nu)^{\dot{\beta}\alpha} = 2g^{\mu\nu}\,, \qquad (\bar{\sigma}^\mu)^{\dot{\alpha}\beta}(\sigma^\nu)_{\beta\dot{\alpha}} = 2g^{\mu\nu}\,. \tag{83}$$

Without proof, we quote the result of contracting the Lorentz indices of a "bared" and an "unbared" Pauli matrix (one may straightforwardly prove it by employing the representation of the Pauli matrices explicitly):

$$(\sigma^\mu)_{\alpha\dot{\beta}}(\bar{\sigma}_\mu)^{\dot{\gamma}\delta} = 2\delta_\alpha^\delta\delta_{\dot{\beta}}^{\dot{\gamma}}\,. \tag{84}$$

Show

$$(\sigma^\mu)_{\alpha\dot{\alpha}}(\sigma_\mu)_{\beta\dot{\beta}} = -2\epsilon_{\alpha\beta}\bar{\epsilon}_{\dot{\alpha}\dot{\beta}}\,. \tag{85}$$

Contractions of products of Pauli matrices with spinors are the bread and butter of calculations in supersymmetric field theories. Hence, quite a bit of gymnastics with such constructions are in order. By the way, showing the validity of the following relations will not be mere exercise. All of them will be needed in relevant calculations later on.

For the anticommuting Weyl spinors ξ^α, $\bar{\lambda}^{\dot{\alpha}}$, θ^α, $\bar{\theta}^{\dot{\alpha}}$, ϕ^α, $\bar{\chi}^{\dot{\alpha}}$, show the following relations:

$$\bar{\epsilon}^{\dot{\alpha}\dot{\beta}}\xi^\alpha(\sigma^\mu)_{\alpha\dot{\beta}} = (\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\xi_\beta =: {}^{\dot{\alpha}}(\bar{\sigma}^\mu\xi)\,, \tag{86}$$

$$(\bar{\lambda}\bar{\sigma}_\mu\xi) = -(\xi\sigma_\mu\bar{\lambda})\,, \tag{87}$$

$$(\theta\xi)(\bar{\theta}\bar{\lambda}) = \frac{1}{2}(\theta\sigma^\mu\bar{\theta})(\xi\sigma_\mu\bar{\lambda})\,, \tag{88}$$

$$(\phi\theta)\bar{\theta}_{\dot{\alpha}} = \frac{1}{2}(\theta\sigma^\mu\bar{\theta})(\phi\sigma_\mu)_{\dot{\alpha}}\,, \tag{89}$$

$$(\theta\sigma^\mu\bar{\xi})(\bar{\theta}\partial_\mu\bar{\chi}) = -\frac{1}{2}(\theta\sigma^\mu\bar{\theta})(\partial_\nu\bar{\chi}\bar{\sigma}_\mu\sigma^\nu\bar{\xi})\,. \tag{90}$$

Now, with most of the previous exercises successfully completed, we should, at last, be reasonably prepared for discussing the building blocks of supersymmetric field theory.

3 Superspace & Superfields

3.1 Superspace

An element of the super-Poincaré group (s$\mathcal{P}$) can be written as

$$g(x,\theta,\bar\theta,\omega) = \underbrace{e^{i\{x_\mu P^\mu + \theta^\alpha Q_\alpha + \bar Q_{\dot\alpha}\bar\theta^{\dot\alpha}\}}}_{t(x,\theta,\bar\theta)\ \text{super translation}} \quad \underbrace{e^{\frac{i}{2}\{\omega^{\mu\nu}M_{\mu\nu}\}}}_{l(\omega)\ \text{Lorentz transformation}} \quad . \tag{91}$$

The t's and the l's form subgroups of s$\mathcal{P}$: s$\mathcal{T}$ and $\mathcal{L}$. The super translations s$\mathcal{T}$ even form an *invariant subgroup*, i.e. $\forall_{g\in s\mathcal{P},t\in s\mathcal{T}}\ :\ g\circ t\circ g^{-1}\ \in\ s\mathcal{T}$ The space parametrised by the translation parameters $z=(x,\theta,\bar\theta)$ is called *superspace* (SSP). Multiplication of a translation $t(z)$ with an arbitrary $g\in s\mathcal{P}$ can always be expressed in terms of a product of t and l:

$$g\circ t(z) = t(z')\circ l(\omega) \ \text{ with }\ z' = z'(g,z), \omega = \omega(g,z)\,. \tag{92}$$

In other words, left multiplication of a super translation with a group element g induces a transformation on the translation parameters z. These induced transformations:

$$g : \begin{cases} \text{SSP} & \to \quad \text{SSP} \\ z & \mapsto \quad z'(g,z) \end{cases} \tag{93}$$

form a representation of s$\mathcal{P}$. This goes under the name coset space construction SSP = s$\mathcal{P}/\mathcal{L}$.

A particularly important (and simple) subclass of induced transformations are multiplications with another super translation:

$$t(x',\theta',\bar\theta') = t(y,\xi,\bar\xi)t(x,\theta,\bar\theta) = e^A e^B \ \text{ with } \begin{cases} A = i(y\cdot P + \xi Q + \bar Q\bar\xi) \\ B = i(x\cdot P + \theta Q + \bar Q\bar\theta) \end{cases} \quad . \tag{94}$$

> Calculate this induced transformation on $z=(x,\theta,\bar\theta)$, i.e. use the Baker-Campbell-Hausdorff Formula, Eq. (3), to write e^Ae^B again as an exponentiation of an element of the algebra and then identify the parameters x'^μ, θ'^α and $\bar\theta'^{\dot\alpha}$ as functions of $(y,\xi,\bar\xi)$ and $(x,\theta,\bar\theta)$.

Due to the Grassmann nature of the spinor parameters $\xi,\bar\xi$ and $\theta,\bar\theta$, the series in the exponent of the left-hand-side of Eq. (3) collapses to two terms and we get for the induced transformations:

$$\begin{aligned} x'^\mu &= x^\mu + y^\mu + i\xi\sigma^\mu\bar\theta - i\theta\sigma^\mu\bar\xi\,, \\ \theta'^\alpha &= \theta^\alpha + \xi^\alpha\,, \\ \bar\theta'^{\dot\alpha} &= \bar\theta^{\dot\alpha} + \bar\xi^{\dot\alpha}\,. \end{aligned} \tag{95}$$

The important special class of transformations where only the Grassmann-valued parameters ξ and $\bar{\xi}$ of the super translations are non-zero is called *supersymmetry transformations*:

$$\begin{aligned} x'^{\mu} &= x^{\mu} + i\xi\sigma^{\mu}\bar{\theta} - i\theta\sigma^{\mu}\bar{\xi}\,, \\ \theta'^{\alpha} &= \theta^{\alpha} + \xi^{\alpha}\,, \\ \bar{\theta}'^{\dot{\alpha}} &= \bar{\theta}^{\dot{\alpha}} + \bar{\xi}^{\dot{\alpha}}\,. \end{aligned} \tag{96}$$

3.2 Superfields

A straightforward way to formulate supersymmetric field theories goes via the introduction of superfields, which are quantum fields in superspace, i.e. operators in Fock space which are functions of the superspace coordinates, $\hat{\Phi}_n(z)$, with a certain Lorentz structure indicated by the index n.

The dependence of $\hat{\Phi}_n$ on the superspace coordinates z can be generated by acting with a super translation transformation on $\hat{\Phi}_n(0,0,0)$:

$$\hat{\Phi}_n(z) = t(z)\hat{\Phi}_n(0,0,0)t^{-1}(z)\,. \tag{97}$$

This formal relation looks almost deceptively simple. However, note that, e.g. if the algebra is represented by differential operators in superspace (as we will introduce shortly), the knowledge of $\hat{\Phi}_n$ and all its first and higher derivatives at $z = (0,0,0)$ are required in order to evaluate the right-hand-side.

The general transformation behaviour of a superfield $\hat{\Phi}_n(z)$ under a super Poincaré transformation is:

$$g\hat{\Phi}_n(z)g^{-1} \equiv t(z_g)l(\omega_g)\hat{\Phi}_n(z)l^{-1}(\omega_g)t^{-1}(z_g) = D^{-1}_{nm}(l)\hat{\Phi}_m(z') \tag{98}$$

with $D_{nm}(l)$ a matrix representation of the Lorentz group and $z' = z'(g,z)$ of Eq. (93).

The transformation behaviour under supersymmetry transformations follows from Eqs. (94) and (97):

$$t(z_g)\hat{\Phi}_n(z)t^{-1}(z_g) = \underbrace{t(z_g)t(z)}_{t(z'(g,z))}\hat{\Phi}_n(0,0,0)\underbrace{t^{-1}(z)t^{-1}(z_g)}_{t^{-1}(z'(g,z))} = \hat{\Phi}_n(z') \tag{99}$$

$$\text{with}\;\; z' = \left\{ \begin{array}{l} x'^{\mu} = x^{\mu} + i\xi\sigma^{\mu}\bar{\theta} - i\theta\sigma^{\mu}\bar{\xi} \\ \theta'^{\alpha} = \theta^{\alpha} + \xi^{\alpha} \\ \bar{\theta}'^{\dot{\alpha}} = \bar{\theta}^{\dot{\alpha}} + \bar{\xi}^{\dot{\alpha}} \end{array} \right. .$$

Scalar Superfield

In a way, the simplest superfield is one which behaves as a scalar under Lorentz transformations, i.e. one where there is no need for the index n used in the previous equations. For the *scalar superfield* $\hat{\Phi}(z)$, its transformation under any super Poincaré transformation g is just the superfield at the transformed coordinates z':

$$g\,\hat{\Phi}(z)\,g^{-1} = \hat{\Phi}(z')\,. \tag{100}$$

$\hat{\Phi}(z)$ can be expressed in terms of an expansion in θ and $\bar{\theta}$ with coefficients which are functions of the space-time coordinates x, the so-called component fields:

$$\begin{aligned}\hat{\Phi}(z) = & f(x) + \theta^\alpha \phi_\alpha(x) + \bar{\theta}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}(x) + (\theta\theta) m(x) + (\bar{\theta}\bar{\theta}) n(x) \\ & + (\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}}) v_\mu(x) + (\theta\theta) \bar{\theta}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}(x) + (\bar{\theta}\bar{\theta}) \theta^\alpha \psi_\alpha(x) + (\theta\theta)(\bar{\theta}\bar{\theta}) d(x) \,. \end{aligned} \tag{101}$$

The component fields consist of four scalar fields (f, m, n, d), one vector field v_μ and four spinor fields (ϕ_α, $\bar{\chi}^{\dot{\alpha}}$, $\bar{\lambda}^{\dot{\alpha}}$, ψ_α). Assuming that all component fields are complex-valued, we count 16 bosonic and 16 fermionic degrees of freedom and exactly half as much if they are real-valued. Clearly, the transformation $z \mapsto z'(g, z)$ in $\Phi(z)$ will not lead out of the space spanned by 1, θ^α, $\bar{\theta}^{\dot{\alpha}}$ and all higher non-vanishing powers of them.

Supersymmetry Transformation of Component Fields

Next, we study the effect of a supersymmetry transformation of a scalar superfield on its component fields.

In order to calculate the transformation properties of component fields of a scalar superfield under a supersymmetry transformation with infinitesimal parameters ξ^α and $\bar{\xi}^{\dot{\alpha}}$, we insert the transformation $z \mapsto z'$ of Eq. (96) in the component field representation in Eq. (101) and perform a Taylor expansion of $\hat{\Phi}(z')$ up to linear order in ξ^α and $\bar{\xi}^{\dot{\alpha}}$.

More explicitly, this is what one should do here:

1. Taylor expand the transformed superfield $\hat{\Phi}(z')$

$$\hat{\Phi}(z') = \hat{\Phi}(z + \delta z(\xi, \bar{\xi})) = \hat{\Phi}(x^\mu + y^\mu, \theta^\alpha + \xi^\alpha, \bar{\theta}^{\dot{\alpha}} + \bar{\xi}^{\dot{\alpha}})$$

up to linear order in the infinitesimal quantities ξ^α and $\bar{\xi}^{\dot{\alpha}}$. (Note the short-hand $y^\mu := i\xi\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\xi}$.) Under this transformation (almost) every term in the component field representation experiences two types of variation: a shift in the θ and $\bar{\theta}$ variables and a shift in the space-time arguments of the component fields. Although normally one has to be very careful Taylor expanding in Grassmann variables, we can use here that for the infinitesimal shift of the space-time argument of any function $F(x)$ by y^μ, the normal Taylor formula holds:

$$F(x+y) = F(x) + \frac{\partial F(x)}{\partial x^\mu} y^\mu \, .$$

One arrives at

$$\hat{\Phi}(z') = \hat{\Phi}(z) + \partial_\mu f(x) y^\mu + \theta^\alpha \partial_\mu \phi_\alpha(x) y^\mu + \xi^\alpha \phi_\alpha + \dots$$

2. After having introduced all shifts, one needs to collect together all terms which multiply the different non-vanishing monomials in θ^α and $\bar{\theta}^{\dot{\alpha}}$. The result can then be identified with the infinitesimal variations of the component fields defined by

$$\hat{\Phi}(z') =: \hat{\Phi}(z) + \delta f(x) + \theta^\alpha \delta\phi_\alpha(x) + \dots$$

In this process, one needs all the spinor relations mentioned above in order to write the monomials in θ^α and $\bar{\theta}^{\dot{\alpha}}$ in the standard form of Eq. (101).

Depending on the time constraints one should try to calculate at least some of the field variations. In particular, $\delta f(x)$ is easy to calculate. Also $\delta d(x)$ is simple to get and it will play an important role later.

The full set of infinitesimal component field variations under a supersymmetry transformation reads:

$$
\begin{aligned}
\delta f(x) &= \xi\phi(x) + \bar{\xi}\bar{\chi}(x)\,, \\
\delta\phi_\alpha(x) &= 2\,\xi_\alpha\, m(x) + {}_\alpha(\sigma^\mu\bar{\xi})[v_\mu(x) - i\partial_\mu f(x)]\,, \\
\delta\bar{\chi}^{\dot{\alpha}}(x) &= 2\,\bar{\xi}^{\dot{\alpha}}\, n(x)\; - {}^{\dot{\alpha}}(\bar{\sigma}^\mu\xi)[v_\mu(x) + i\partial_\mu f(x)]\,, \\
\delta m(x) &= \bar{\xi}\bar{\lambda}(x) + \frac{i}{2}(\partial_\mu\phi(x)\sigma^\mu\bar{\xi})\,, \\
\delta n(x) &= \xi\psi(x) - \frac{i}{2}(\xi\sigma^\mu\partial_\mu\bar{\chi}(x))\,, \\
\delta v_\mu(x) &= \xi\sigma_\mu\bar{\lambda}(x) + \psi(x)\sigma_\mu\bar{\xi} - \frac{i}{2}(\xi\sigma_\nu\bar{\sigma}_\mu\,\partial^\nu\phi(x)) + \frac{i}{2}(\partial^\nu\bar{\chi}(x)\,\bar{\sigma}_\mu\sigma_\nu\bar{\xi})\,, \\
\delta\bar{\lambda}^{\dot{\alpha}}(x) &= 2\,\bar{\xi}^{\dot{\alpha}}\, d(x) - i^{\dot{\alpha}}(\bar{\sigma}^\mu\xi)\,\partial_\mu m(x) - \frac{i}{2}{}^{\dot{\alpha}}(\bar{\sigma}^\mu\sigma^\nu\bar{\xi})\,\partial_\nu v_\mu(x)\,, \\
\delta\psi_\alpha(x) &= 2\,\xi_\alpha d(x) - i_\alpha(\sigma^\mu\bar{\xi})\,\partial_\mu n(x) + \frac{i}{2}{}_\alpha(\sigma^\mu\bar{\sigma}^\nu\xi)\,\partial_\nu v_\mu(x)\,, \\
\delta d(x) &= \partial_\mu\left(-\frac{i}{2}\xi\sigma^\mu\bar{\lambda}(x) + \frac{i}{2}\psi(x)\sigma^\mu\bar{\xi}\right).
\end{aligned}
\tag{102}
$$

(103)

Note that the "highest" component of the scalar superfield (i.e. the $(\theta\theta)(\bar{\theta}\bar{\theta})$ component), $d(x)$, transforms as a total derivative. This feature will be employed below when we show how actions invariant under supersymmetry transformations can be constructed.

3.3 Superspace Representations of SUSY Generators

Given a scalar function of the superspace coordinates z, i.e. a scalar superfield $\hat{\Phi}(z)$, we can derive a representation of the generators of the super translations, $\hat{P}$, $\hat{Q}$ and $\hat{\bar{Q}}$, as differential operators acting on functions in superspace:

$$
\hat{\Phi}(z') = e^{i(y\cdot P + \xi Q + \bar{Q}\bar{\xi})}\hat{\Phi}(z)e^{-i(y\cdot P + \xi Q + \bar{Q}\bar{\xi})} =: e^{-i(y\cdot\hat{P} + \xi\hat{Q} + \hat{\bar{Q}}\bar{\xi})}\hat{\Phi}(z)
\tag{104}
$$

> Derive the differential operators $\hat{P}_\mu$, $\hat{Q}_\alpha$ and $\hat{\bar{Q}}_{\dot{\alpha}}$. In order to do so, it is enough to study infinitesimal transformation parameters y^μ, ξ^α and $\bar{\xi}^{\dot{\alpha}}$. Taylor expanding the left-most and right-most positions of the above equation chain should do the trick.

The result is

$$
\hat{P}_\mu = i\partial_\mu\,, \qquad \hat{Q}_\alpha = i\frac{\partial}{\partial\theta^\alpha} - {}_\alpha(\sigma^\mu\bar{\theta})\frac{\partial}{\partial x^\mu}\,, \qquad \hat{\bar{Q}}_{\dot{\alpha}} = -i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + (\theta\sigma^\mu)_{\dot{\alpha}}\frac{\partial}{\partial x^\mu}\,.
\tag{105}
$$

Check the validity of the essential supersymmetry relation, Eq. (22), for the representation in terms of differential operators just derived, i.e. show

$$\{\hat{Q}_\alpha, \hat{\bar{Q}}_{\dot{\alpha}}\} = 2\,\sigma^\mu_{\alpha\dot{\alpha}} \hat{P}_\mu\,.$$

In what follows, we will always work in superspace and use the representations of supersymmetry generators as differential operators in superspace. Hence, we will drop the hat symbol ($\hat{}$) from now on, assuming implicitly that all operators mentioned are understood to be differential operators in superspace.

3.4 SUSY-Covariant Derivative

Differential operators which anti-commute with supersymmetry transformations allow to formulate constraint equations which are invariant under supersymmetry transformations. Such operators D_α and $\bar{D}_{\dot{\alpha}}$ have to fulfil:

$$\{D_\alpha, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0\,. \tag{106}$$

Use the ansatz of a linear differential operator

$$D_\alpha = \partial_\alpha + v \cdot {}_\alpha(\sigma^\mu \bar{\theta})\partial_\mu\,, \quad v \in \mathbf{C}\,,$$

and show that for $v = -i$ all required properties in Eq. (106) are fulfilled. In the process, determine the corresponding expression for $\bar{D}_{\dot{\alpha}}$ by complex conjugation (Note the Grassmann subtleties!). Finally, calculate the anti-commutator $\{D_\alpha, \bar{D}_{\dot{\alpha}}\}$.

Let's collect the results here:

$$D_\alpha = \quad \partial_\alpha - i_\alpha(\sigma^\mu \bar{\theta})\partial_\mu\,, \tag{107}$$
$$\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} + i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu\,, \tag{108}$$

and

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = 2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\,. \tag{109}$$

The linear differential operators D_α and $\bar{D}_{\dot{\alpha}}$ are called the SUSY-covariant derivatives. They also anti-commute among themselves

$$\{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0\,. \tag{110}$$

From their form and properties in Eq. (106), it follows that

$$[P_\mu, D_\alpha] = [P_\mu, \bar{D}_{\dot\alpha}] = [\xi Q, D_\alpha] = [\xi Q, \bar{D}_{\dot\alpha}] = [\bar{Q}\bar{\xi}, D_\alpha] = [\bar{Q}\bar{\xi}, \bar{D}_{\dot\alpha}] = 0\,.$$

Hence,

$$e^{-i(y\cdot\hat{P} + \xi\hat{Q} + \hat{\bar{Q}}\bar{\xi})} D_\alpha = D_\alpha e^{-i(y\cdot\hat{P} + \xi\hat{Q} + \hat{\bar{Q}}\bar{\xi})}\,,$$

and likewise for $\bar{D}_{\dot\alpha}$. So, indeed, constraint equations on a superfield which are formulated entirely with SUSY-covariant derivatives acting on the superfield are invariant under supersymmetry transformations (in fact even under super translations). Requiring a superfield to fulfil such a constraint equation will lead to a smaller representation space of supersymmetry than the unconstrained superfield furnishes. This is the way to obtain irreducible representations of the supersymmetry algebra in the superspace formalism.

3.5 Restricted Superfields

We will walk through three major ways how to constrain a general scalar superfield $\hat\Phi(z)$ in order to obtain smaller representation spaces: $D_\alpha\hat\Phi(z) = 0$, $\bar{D}_{\dot\alpha}\hat\Phi(z) = 0$ and the reality condition $\hat\Phi^\dagger(z) = \Phi(z)$

The Chiral Superfield

The chiral superfield fulfils the condition

$$\bar{D}_{\dot\alpha}\hat\Phi(z) = 0\,. \tag{111}$$

Use the component field decomposition of a general scalar superfield in Eq. (101) and the definition of $\bar{D}_{\dot\alpha}$ in Eq. (108) to calculate the restrictions resulting from Eq. (111) on the component field and calculate the component field decomposition of $\hat\Phi(z)|_{\text{chiral}}$ using those restrictions on $\hat\Phi(z)$.

Eq. (111) leads to the following constraints on the component fields of a general scalar superfield:

$$\bar\chi_{\dot\alpha}(x) = 0\,, \qquad n(x) = 0\,, \qquad \psi_\alpha(x) = 0\,, \tag{112}$$

$$v_\mu(x) = -i\partial_\mu f(x)\,, \qquad \bar\lambda_{\dot\alpha}(x) = \frac{i}{2}(\partial_\mu\phi(x)\sigma^\mu)_{\dot\alpha}\,, \qquad d(x) = -\frac{1}{4}\Box f(x)\,. \tag{113}$$

Hence, we have

$$\hat\Phi(z)|_{\text{chiral}} = f(x) + \theta^\alpha\phi_\alpha(x) + (\theta\theta)m(x) - i(\theta\sigma^\mu\bar\theta)\partial_\mu f(x) \tag{114}$$

$$+ \frac{i}{2}(\theta\theta)(\partial_\mu\phi(x)\sigma^\mu\bar\theta) - \frac{1}{4}(\theta\theta)(\bar\theta\bar\theta)\Box f(x)\,. \tag{115}$$

The usual naming convention (and normalisation) for the chiral superfield an its components is a bit different to what we wrote. To obtain it, the following shifts of symbols is needed:

$$\hat{\Phi}(z)|_{\text{chiral}} \to \hat{\phi}(z)\,, \qquad f(x) \to A(x)\,, \qquad \phi_\alpha(x) \to \sqrt{2}\psi_\alpha(x)\,, \qquad m(x) \to F(x)\,. \tag{116}$$

With this shift in convention, we get

$$\begin{aligned}\hat{\phi}(z) = A(x) + \sqrt{2}\theta^\alpha\psi_\alpha(x) + (\theta\theta)F(x) - i(\theta\sigma^\mu\bar{\theta})\partial_\mu A(x) \\ + \frac{i}{\sqrt{2}}(\theta\theta)(\partial_\mu\psi(x)\sigma^\mu\bar{\theta}) - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\Box A(x)\,,\end{aligned} \tag{117}$$

which is the general solution to the supersymmetric constraint equation

$$\bar{D}_{\dot{\alpha}}\hat{\phi}(z) = 0\,. \tag{118}$$

Note that the independent component with the highest power in the polynomial in θ and $\bar{\theta}$ is the $(\theta\theta)$ component $F(x)$. All higher components are determined by lower ones through the acting of derivatives and the $(\bar{\theta}\bar{\theta})$ as well as the $(\bar{\theta}\bar{\theta})\theta_\alpha$ component vanish. Clearly, the representation space of supersymmetry which the chiral superfield furnishes is of lower dimension than of the unconstrained scalar superfield: there are two complex scalar fields $A(x)$ and $F(x)$ with together four real bosonic field degrees of freedom and a complex two-component Weyl spinor field $\psi_\alpha(x)$ with matching four additional real fermionic degrees of freedom.

Although this should be clear by construction, we would like to show now explicitly that the component fields $A(x)$, $F(x)$ and $\psi_\alpha(x)$ transform among themselves under supersymmetry transformations. For this we can employ our previously derived formula for the transformation property of the general scalar superfield, Eq. (102).

Employ the previously derived formula for the transformation property of the general scalar superfield, Eq. (102), using the naming convention of Eq. (116) to show the transformation properties of the component fields of chiral superfield under an infinitesimal supersymmetry transformation:

$$\begin{aligned}\delta A(x) &= \sqrt{2}\xi^\alpha\psi_\alpha(x)\,, \\ \delta\psi_\alpha(x) &= \sqrt{2}\xi_\alpha F(x) - i\sqrt{2}_\alpha(\sigma^\mu\bar{\xi})\partial_\mu A(x)\,, \\ \delta F(x) &= \partial_\mu(i\sqrt{2}\psi(x)\sigma^\mu\bar{\xi})\,.\end{aligned} \tag{119}$$

Similar to the general scalar superfield, the "highest" independent component of the chiral superfield (i.e. the $(\theta\theta)$ component), $F(x)$, transforms again as a total derivative.

The Anit-Chiral Superfield

The so-called anti-chiral superfield is a scalar superfield constrained by the condition

$$D_\alpha \hat{\Phi}(z) = 0\,. \tag{120}$$

We can work out the solution to this equation in the same way as in the previous section. However, as D_α and $\bar{D}_{\dot\alpha}$ are related by complex conjugation, so are the solutions to the conditions. Hence, the complex conjugate of a chiral superfield $\hat{\phi}(z)$ is the general solution to the condition for the anti-chiral superfield:

$$\begin{aligned}\hat{\phi}^\dagger(z) = A^\star(x) + \sqrt{2}\bar{\theta}_{\dot\alpha}\bar{\psi}^{\dot\alpha}(x) + (\bar{\theta}\bar{\theta})F^\star(x) + i(\theta\sigma^\mu\bar{\theta})\partial_\mu A^\star(x) \\ - \frac{i}{\sqrt{2}}(\bar{\theta}\bar{\theta})(\theta\sigma^\mu\partial_\mu\bar{\psi}(x)) - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\Box A^\star(x)\,.\end{aligned} \tag{121}$$

Do the complex conjugation exercise, i.e. determine $\hat{\phi}^\dagger(z)$ from $\phi(z)$ in Eq. (117).

In terms of our original naming convention for the component fields in Eq. (101), which is related to the one above via

$$\hat{\Phi}(z)|_{\text{anti-chiral}} \to \hat{\phi}^\dagger(z)\,, \quad f(x) \to A^\star(x)\,, \quad \bar{\chi}_{\dot\alpha}(x) \to \sqrt{2}\bar{\psi}_{\dot\alpha}(x)\,, \quad n(x) \to F^\star(x)\,, \tag{122}$$

the condition (120) implies for the components of a general scalar superfield:

$$\phi_\alpha(x) = 0\,, \qquad m(x) = 0\,, \qquad \bar{\lambda}_{\dot\alpha}(x) = 0\,, \tag{123}$$

$$v_\mu(x) = i\partial_\mu f(x)\,, \qquad \psi_\alpha(x) = -\frac{i}{2}{}_\alpha(\sigma^\mu\partial_\mu\bar{\chi}(x))\,, \qquad d(x) = -\frac{1}{4}\Box f(x)\,. \tag{124}$$

The Real Superfield (Vector Superfield)

The real superfield is defined by the condition

$$\hat{\Phi}^\dagger(z) = \hat{\Phi}(z)\,. \tag{125}$$

Using Eq. (101) and its complex conjugate, we get the following conditions for the component fields:

$$\begin{aligned} &f^\star(x) = f(x)\,, \quad \chi_\alpha(x) = \phi_\alpha\,, \quad m^\star(x) = n(x)\,, \quad n^\star(x) = m(x)\,, \\ &v^\star_\mu(x) = v_\mu(x)\,, \quad \lambda_\alpha(x) = \psi_\alpha(x)\,, \quad d^\star(x) = d(x)\,.\end{aligned} \tag{126}$$

The number of bosonic and fermionic degrees of freedom are both exactly halved compared to the general scalar superfield, i.e. there are now 8 bosonic and 8 fermionic degrees of freedom. In particular, in contrast to the chiral and anti-chiral superfields, a spin-1 or vector field is part of the component fields here. Real superfields allow for a supersymmetric generalisation of gauge vector fields and are therefore also often called "vector superfields."

In order to illustrate its function as the supersymmetric generalisation of a gauge field, a reparametrisation (and some slight change in notation) is in order:

$$\hat{\Phi}(z)|_{\text{real}} \to V(z)\,, \tag{127}$$

$$f(x) \to C(x) \in \mathbf{R}\,, \tag{128}$$

$$\phi_\alpha(x) \to -i\chi_\alpha(x)\,, \tag{129}$$

$$\bar{\chi}^{\dot{\alpha}} \to i\bar{\chi}^{\dot{\alpha}}\,, \tag{130}$$

$$m(x) \to -\frac{i}{2}(M(x) - iN(x))\,, \quad M, N \in \mathbf{R}\,, \tag{131}$$

$$n(x) \to \frac{i}{2}(M(x) + iN(x))\,, \tag{132}$$

$$v_\mu(x) \to -v_\mu(x)\,, \tag{133}$$

$$\bar{\lambda}^{\dot{\alpha}}(x) \to -i(\bar{\lambda}^{\dot{\alpha}}(x) - \frac{i}{2}{}^{\dot{\alpha}}(\bar{\sigma}^\mu \partial_\mu \chi(x)))\,, \tag{134}$$

$$\psi_\alpha(x) \to i(\lambda_\alpha(x) - \frac{i}{2}{}_\alpha(\sigma^\mu \partial_\mu \bar{\chi}(x))))\,, \tag{135}$$

$$d(x) \to -\frac{1}{2}(D(x) + \frac{1}{2}\Box C(x))\,, \quad D \in \mathbf{R}\,. \tag{136}$$

The real or vector superfield then reads

$$\begin{aligned}\hat{V}(z) = {} & C(x) - i\theta\chi(x) + i\bar{\theta}\bar{\chi}(x) - \frac{i}{2}(\theta\theta)[M(x) - iN(x)] + \frac{i}{2}(\bar{\theta}\bar{\theta})[M(x) + iN(x)] \\ & - (\theta\sigma^\mu\bar{\theta})v_\mu(x) - i(\theta\theta)\bar{\theta}_{\dot{\alpha}}[\bar{\lambda}^{\dot{\alpha}}(x) - \frac{i}{2}{}^{\dot{\alpha}}(\bar{\sigma}^\mu\partial_\mu\chi(x))] + i(\bar{\theta}\bar{\theta})\theta^\alpha[\lambda_\alpha(x) - \frac{i}{2}{}_\alpha(\sigma^\mu\partial_\mu\bar{\chi}(x))] \\ & - \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})[D(x) + \frac{1}{2}\Box C(x)]\,.\end{aligned} \tag{137}$$

A real superfield $\hat{V}(z)$ can be used as a supersymmetric generalisation of a gauge field. In the Abelian case, the gauge transformation of a normal gauge field $v_\mu(x)$,

$$v_\mu(x) \mapsto v_\mu(x) + \frac{1}{g}\partial_\mu\alpha(x)\,,$$

with some arbitrary function $\alpha(x)$, generalises in the supersymmetric case to

$$\hat{V}(z) \mapsto \hat{V}(z) + \hat{\phi}(z) + \hat{\phi}^\dagger(z)\,, \tag{138}$$

with an arbitrary chiral superfield $\hat{\phi}(z)$ and its corresponding, anti-chiral, complex conjugate superfield $\hat{\phi}^\dagger(z)$. Later, when we will discuss gauge interactions in more detail, we will use the notation $\hat{\phi}(z) \to -i\Lambda(z)/(2g)$ and $\hat{\phi}^\dagger(z) \to i\Lambda^\dagger(z)/(2g)$.

Calculate $\hat{\phi}(z) + \hat{\phi}^\dagger(z)$ in component fields, compare with the component field representation $\hat{V}(z)$ in Eq. (137) and read off the super gauge transformations for the component fields of $\hat{V}(z)$.

The behaviour of the component fields of a vector superfield under a super gauge transformation are as follows (the sense in the awkward-looking convention we chose for the component fields should now be apparent):

$$\begin{aligned}
C(x) &\mapsto C(x) + 2\mathrm{Re}[A(x)]\,, \\
\chi_\alpha(x) &\mapsto \chi_\alpha(x) + i\sqrt{2}\psi_\alpha(x)\,, \\
\bar{\chi}^{\dot{\alpha}}(x) &\mapsto \bar{\chi}^{\dot{\alpha}}(x) - i\sqrt{2}\bar{\psi}^{\dot{\alpha}}(x)\,, \\
M(x) - iN(x) &\mapsto M(x) - iN(x) + 2iF(x)\,, \\
M(x) + iN(x) &\mapsto M(x) + iN(x) - 2iF^\star(x)\,, \\
v_\mu(x) &\mapsto v_\mu(x) + \partial_\mu(-2\mathrm{Im}[A(x)])\,, \\
\lambda^\alpha(x) &\mapsto \lambda^\alpha(x)\,, \\
D(x) &\mapsto D(x)\,.
\end{aligned} \tag{139}$$

It is easy to see that the fields C, M, N and χ can be eliminated completely by a choice of $\mathrm{Re}(A)$, F and ψ. While this choice alone does not fix a gauge completely, it is an important gauge class, called *Wess Zumino gauge* after its inventors. The normal gauge freedom of the vector field v_μ remains untouched by this choice. However, the Wess Zumino gauge breaks invariance under supersymmetry transformations. Note that the component fields λ and D do not change at all under a super gauge transformation, i.e. they are super gauge invariant. While this insight looks rather trivial here, it will become a major tool in the construction of gauge invariant and supersymmetric interaction Lagrangians, once we have shown that products of superfields are again superfields.

Calculate $\hat{V}(z)$ in Wess Zumino gauge.

The result of this exercise is:

$$\hat{V}(z)|_{\mathrm{WZ}} = -(\theta\sigma^\mu\bar{\theta})v_\mu(x) - i(\theta\theta)\bar{\theta}\bar{\lambda}(x) + i(\bar{\theta}\bar{\theta})\theta\lambda(x) - \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})D(x)\,. \tag{140}$$

Most importantly, the vector superfield in Wess Zumino gauge has only components with at least two Grassmann factors attached. Hence, already the third power of it vanishes:

$$(\hat{V}(z)|_{\mathrm{WZ}})^3 = 0\,. \tag{141}$$

In a supersymmetric gauge theory, the fermionic components $\lambda(x)$ and $\bar{\lambda}(x)$ comprise the spin-$\frac{1}{2}$ superpartner to the gauge field $v_\mu(x)$, giving rise to the particles called gauginos.

The scalar field $D(x)$ turns out to be an auxiliary degree of freedom, with no dynamical equation of motion, i.e. there is no kinetic term for $D(x)$ in the Lagrangian. It can be eliminated using its equation of motion which relates it algebraically to physical degrees of freedom.

3.6 Products of Superfields

Our ultimate goal in this course is to formulate a supersymmetric model of interacting particles. Interactions in quantum field theory are described by products of local operators. Hence, we are going to study products of superfields next. It is very helpful in that respect, as we will show below, that products of superfields are again superfields, and, in particular, that the product of all chiral or all anti-chiral factors will also be a chiral or anti-chiral superfield, respectively.

Products of Superfields are Superfields

Given two general scalar superfields $\hat{\Phi}_1(z)$ and $\hat{\Phi}_2(z)$, we would like to study the basic properties of the product

$$\hat{\Phi}_{12} := \hat{\Phi}_1 \hat{\Phi}_2 \,.$$

First of all, the field $\hat{\Phi}_{12}$ at given coordinates z in superspace is the result of acting with a super translation on the field at the origin and it should coincide with the product $\hat{\Phi}_1$ and $\hat{\Phi}_2$ at z:

$$\begin{aligned}
\hat{\Phi}_{12}(z) &= t(z)\hat{\Phi}_{12}(0,0,0)t^{-1}(z) \\
&= t(z)\hat{\Phi}_1(0,0,0)\hat{\Phi}_2(0,0,0)t^{-1}(z) \\
&= t(z)\hat{\Phi}_1(0,0,0)t^{-1}(z)\, t(z)\hat{\Phi}_2(0,0,0)t^{-1}(z) \\
&= \hat{\Phi}_1(z)\hat{\Phi}_2(z) \,.
\end{aligned}$$

Clearly, the same chain of operations can be repeated for any element g of the super Poincaré group instead of t acting on $\hat{\Phi}_{12}(z)$, thus showing that the product $\hat{\Phi}_{12}(z)$ transforms as a scalar superfield:

$$g\hat{\Phi}_{12}(z)g^{-1} = \hat{\Phi}_{12}(z') \,.$$

Hence, the product of scalar superfields is itself a superfield, and, as this reasoning can be iterated, any multiple product of scalar superfields is also a scalar superfield.

Component Fields of the Product $\hat{\Phi}_{12}$

Here's now another of the more lengthy calculations using the Weyl spinor formalism: the component fields of a product of two superfields expressed in terms of the component fields of the two factors. It's tedious but it pays off greatly, because once one has the general formula, the product of arbitrary numbers of superfields or special ones is then piece of cake. Let's chose names:

$$\begin{aligned}
\hat{\Phi}_1(z) &= f_1(x) + \theta^\alpha \phi_{1,\alpha}(x) + \bar{\theta}_{\dot{\alpha}} \bar{\chi}_1^{\dot{\alpha}}(x) + (\theta\theta) m_1(x) + (\bar{\theta}\bar{\theta}) n_1(x) + (\theta\sigma^\mu\bar{\theta}) v_{1,\mu}(x) \\
&\quad + (\theta\theta)\bar{\theta}_{\dot{\alpha}} \bar{\lambda}_1^{\dot{\alpha}}(x) + (\bar{\theta}\bar{\theta})\theta^\alpha \psi_{1,\alpha}(x) + (\theta\theta)(\bar{\theta}\bar{\theta}) d_1(x)\,, \qquad (142)\\
\hat{\Phi}_2(z) &= f_2(x) + \theta^\alpha \phi_{2,\alpha}(x) + \bar{\theta}_{\dot{\alpha}} \bar{\chi}_2^{\dot{\alpha}}(x) + (\theta\theta) m_2(x) + (\bar{\theta}\bar{\theta}) n_2(x) + (\theta\sigma^\mu\bar{\theta}) v_{2,\mu}(x) \\
&\quad + (\theta\theta)\bar{\theta}_{\dot{\alpha}} \bar{\lambda}_2^{\dot{\alpha}}(x) + (\bar{\theta}\bar{\theta})\theta^\alpha \psi_{2,\alpha}(x) + (\theta\theta)(\bar{\theta}\bar{\theta}) d_2(x)\,, \qquad (143)\\
\hat{\Phi}_{12}(z) &= f_{12}(x) + \theta^\alpha \phi_{12,\alpha}(x) + \bar{\theta}_{\dot{\alpha}} \bar{\chi}_{12}^{\dot{\alpha}}(x) + (\theta\theta) m_{12}(x) + (\bar{\theta}\bar{\theta}) n_{12}(x) + (\theta\sigma^\mu\bar{\theta}) v_{12,\mu}(x) \\
&\quad + (\theta\theta)\bar{\theta}_{\dot{\alpha}} \bar{\lambda}_{12}^{\dot{\alpha}}(x) + (\bar{\theta}\bar{\theta})\theta^\alpha \psi_{12,\alpha}(x) + (\theta\theta)(\bar{\theta}\bar{\theta}) d_{12}(x)\,. \qquad (144)
\end{aligned}$$

Calculate, at least, $f_{12}(x)$, $\phi_{12,\alpha}(x)$ and $m_{12}(x)$ in terms of component fields of $\hat{\Phi}_1(x)$ and $\hat{\Phi}_2(x)$.

If you are patient enough to do all the spinor "gymnastics" you should arrive at (the x dependence has been dropped in the notation):

$$\begin{aligned}
f_{12} &= f_1 f_2\,,\\
\phi_{12,\alpha} &= f_1\phi_{2,\alpha} + f_2\phi_{1,\alpha}\,,\\
\bar{\chi}_{12}^{\dot{\alpha}} &= f_1\bar{\chi}_2^{\dot{\alpha}} + f_2\bar{\chi}_1^{\dot{\alpha}}\,,\\
m_{12} &= f_1 m_2 + f_2 m_1 - \frac{1}{2}(\phi_1\phi_2)\,,\\
n_{12} &= f_1 n_2 + f_2 n_1 - \frac{1}{2}(\bar{\chi}_1\bar{\chi}_2)\,, \qquad (145)\\
v_{12}^\mu &= f_1 v_2^\mu + f_2 v_1^\mu + \frac{1}{2}\phi_1\sigma^\mu\bar{\chi}_2 + \frac{1}{2}\phi_2\sigma^\mu\bar{\chi}_1\,,\\
\bar{\lambda}_{12}^{\dot{\alpha}} &= f_1\bar{\lambda}_2^{\dot{\alpha}} + f_2\bar{\lambda}_1^{\dot{\alpha}} + m_1\bar{\chi}_2^{\dot{\alpha}} + m_2\bar{\chi}_1^{\dot{\alpha}} + \frac{1}{2}{}^{\dot{\alpha}}(\bar{\sigma}^\mu\phi_1)v_{2,\mu} + \frac{1}{2}{}^{\dot{\alpha}}(\bar{\sigma}^\mu\phi_2)v_{1,\mu}\,,\\
\psi_{12,\alpha} &= f_1\psi_{2,\alpha} + f_2\psi_{1,\alpha} + n_1\phi_{2,\alpha} + n_2\phi_{1,\alpha} - \frac{1}{2}{}_\alpha(\sigma^\mu\bar{\chi}_1)v_{2,\mu} - \frac{1}{2}{}_\alpha(\sigma^\mu\bar{\chi}_2)v_{1,\mu}\,,\\
d_{12} &= f_1 d_2 + f_2 d_1 + m_1 n_2 + m_2 n_1 - \frac{1}{2}\bar{\chi}_1\bar{\lambda}_2 - \frac{1}{2}\bar{\chi}_2\bar{\lambda}_1 - \frac{1}{2}\phi_1\psi_2 - \frac{1}{2}\phi_2\psi_1 + \frac{1}{2}v_{1,\mu}v_2^\mu\,.
\end{aligned}$$

If we had doubts before, whether the product of two scalar superfields is commuting, the results above show it clearly:

$$\hat{\Phi}_1(z)\hat{\Phi}_2(z) = \hat{\Phi}_2(z)\hat{\Phi}_1(z)\,. \qquad (146)$$

Product of two Chiral Superfields

As the SUSY covariant derivative $\bar{D}_{\dot{\alpha}}$ is a linear operator, it is clear that the Leibniz rule applies. Hence, for products of chiral superfields $\hat{\phi}_1$ and $\hat{\phi}_2$ it immediately follows, that the product $\hat{\phi}_{12} = \hat{\phi}_1\hat{\phi}_2$ is also a chiral superfield:

$$\bar{D}_{\dot\alpha}\hat{\phi}_{12} = \bar{D}_{\dot\alpha}(\hat{\phi}_1\hat{\phi}_2) = (\bar{D}_{\dot\alpha}\hat{\phi}_1)\hat{\phi}_2 + \hat{\phi}_1(\bar{D}_{\dot\alpha}\hat{\phi}_2) = 0\,. \tag{147}$$

Then, by induction, it follows that any multiple product of chiral superfields is also a chiral superfield.

Using the component field solution of the chirality condition Eq. (111) for two superfields $\hat{\phi}_1$ and $\hat{\phi}_2$,

$$\bar{\chi}_i = n_i = \psi_i = 0\,,\quad v_i^\mu = -i\partial^\mu f_i\,,\quad \bar{\lambda}_{i,\dot\alpha} = \frac{i}{2}(\partial_\mu\phi_i\sigma^\mu)_{\dot\alpha}\,,\quad d_i = -\frac{1}{4}\Box f_i\quad (i=1,2)\,,$$

specialise our general product formula Eq. (145) to the product of two chiral superfields $\hat{\phi}_{12} = \hat{\phi}_1\hat{\phi}_2$.

Convince yourself that the component fields of the product again fulfil the chirality condition.

Thus we get for the component fields of $\hat{\phi}_{12}$:

$$\begin{aligned}
f_{12} &= f_1 f_2 \\
\phi_{12,\alpha} &= f_1\phi_{2,\alpha} + f_2\phi_{1,\alpha}\,, \\
\bar{\chi}_{12}^{\dot\alpha} &= 0\,, \\
m_{12} &= f_1 m_2 + f_2 m_1 - \frac{1}{2}(\phi_1\phi_2)\,, \\
n_{12} &= 0\,, \\
v_{12}^\mu &= -if_1\partial^\mu f_2 - if_2\partial^\mu f_1 = -i\partial^\mu(f_1f_2)\,, \\
\bar{\lambda}_{12}^{\dot\alpha} &= \frac{i}{2}\partial_\mu\big(f_1(\phi_2\sigma^\mu)_{\dot\alpha} + f_2(\phi_1\sigma^\mu)_{\dot\alpha}\big) = \frac{i}{2}\partial_\mu(\phi_{12}\sigma^\mu)_{\dot\alpha}\,, \\
\psi_{12,\alpha} &= 0\,, \\
d_{12} &= -\frac{1}{4}f_1\Box f_2 - \frac{1}{4}f_2\Box f_1 = -\frac{1}{4}\Box f_{12}\,.
\end{aligned} \tag{148}$$

From the discussion above, we knew already that products of chiral superfields are again chiral superfields. Nevertheless, it is instructive to see, that the component fields of $\hat{\phi}_{12}$ readily resemble the solution to the chirality condition in terms of component fields which we derived in the previous section.

Product of three Chiral Superfields

We saw already that the $(\theta\theta)$ component of a chiral superfield transforms as a total derivative under supersymmetry transformations and it is likewise for the $(\bar\theta\bar\theta)$ component of an anti-chiral superfield. Such terms will be building blocks of supersymmetric Lagrangians.

For this reason, we restrict our attention in the evaluation of the product of three chiral superfields to the $(\theta\theta)$ component of this product.

Use Eq. (148) to determine the $(\theta\theta)$ component of three chiral superfields

$$\hat{\phi}_3(z)\hat{\phi}_2(z)\hat{\phi}_1(z)\big|_{(\theta\theta)} \,.$$

Here the solution:

$$\begin{aligned}
\hat{\phi}_3(z)\hat{\phi}_2(z)\hat{\phi}_1(z)\big|_{(\theta\theta)} &= m_{321}(\theta\theta)\,, \\
\hat{\phi}_3(z)\hat{\phi}_{12}(z)\big|_{(\theta\theta)} &\overset{(148)}{=} f_3 m_{12} + f_{12} m_3 - \frac{1}{2}(\phi_3\phi_{12})\,, \\
&\overset{(148)}{=} f_3(f_1 m_2 + f_2 m_1 - \frac{1}{2}(\phi_2\phi_2)) + f_1 f_2 m_3 - \frac{1}{2}(\phi_3(f_1\phi_2 + f_2\phi_1))\,.
\end{aligned}$$

Hence, we have :

$$\hat{\phi}_3(z)\hat{\phi}_2(z)\hat{\phi}_1(z)\big|_{(\theta\theta)} = f_1 f_3 m_2 + f_2 f_3 m_1 + f_1 f_2 m_3 - \frac{1}{2}f_3(\phi_1\phi_2) - \frac{1}{2}f_1(\phi_3\phi_2) - \frac{1}{2}f_2(\phi_3\phi_1)\,, \tag{149}$$

where the x dependence of the component fields is not shown. Shifting to the conventional naming and normalisation, Eq. (116), we have

$$\hat{\phi}_3(z)\hat{\phi}_2(z)\hat{\phi}_1(z)\big|_{(\theta\theta)} = A_1A_3F_2 + A_2A_3F_1 + A_1A_2F_3 - A_3(\psi_1\psi_2) - A_1(\psi_3\psi_2) - A_2(\psi_3\psi_1)\,. \tag{150}$$

The corresponding formula for the $(\bar{\theta}\bar{\theta})$ component of the product of three anti-chiral superfields $\hat{\phi}_3^\dagger\hat{\phi}_2^\dagger\hat{\phi}_1^\dagger$ can be obtained by complex conjugation:

$$\hat{\phi}_3^\dagger(z)\hat{\phi}_2^\dagger(z)\hat{\phi}_1^\dagger(z)\big|_{(\bar{\theta}\bar{\theta})} = A_1^\star A_3^\star F_2^\star + A_2^\star A_3^\star F_1^\star + A_1^\star A_2^\star F_3^\star - A_3^\star(\bar{\psi}_1\bar{\psi}_2) - A_1^\star(\bar{\psi}_3\bar{\psi}_2) - A_2^\star(\bar{\psi}_3\bar{\psi}_1)\,. \tag{151}$$

Products of Anti-Chiral and Chiral Superfields

With products of two and three chiral or anti-chiral superfields, we have already some building blocks for supersymmetric interaction terms at hand. Yet, we still lack kinetic terms. To that end, we study the product $\hat{\phi}^\dagger\hat{\phi}$ which we know is a real superfield. We also know that the $(\theta\theta)(\bar{\theta}\bar{\theta})$ component of a real superfield transforms as a total derivative under supersymmetry transformations, and thus serves as another potential building block of supersymmetric Lagrangians.

Calculate

$$\hat{\phi}^\dagger \hat{\phi}\big|_{(\theta\theta)(\bar{\theta}\bar{\theta})}$$

using our general formula for products of two superfields Eq. (145) with

$$\hat{\Phi}_1 = \hat{\phi}^\dagger : \begin{cases} f_1 \rightarrow A^\star, & \phi_{1,\alpha} \rightarrow 0, & \bar{\chi}_{1,\dot{\alpha}} \rightarrow \sqrt{2}\bar{\psi}_{\dot{\alpha}}, \\ m_1 \rightarrow 0, & n_1 \rightarrow F^\star, & v_{1,\mu} \rightarrow i\partial_\mu A^\star, \\ \bar{\lambda}_1^{\dot{\alpha}} \rightarrow 0, & \psi_{1,\alpha} \rightarrow -\frac{i}{\sqrt{2}}{}_\alpha(\sigma^\mu \partial_\mu \bar{\psi}), & d_1 \rightarrow -\frac{1}{4}\Box A^\star \end{cases} ,$$

$$\hat{\Phi}_2 = \hat{\phi} : \begin{cases} f_2 \rightarrow A, & \phi_{2,\alpha} \rightarrow \sqrt{2}\psi_\alpha, & \bar{\chi}_{2,\dot{\alpha}} \rightarrow 0, \\ m_2 \rightarrow F, & n_2 \rightarrow 0, & v_{2,\mu} \rightarrow -i\partial_\mu A, \\ \bar{\lambda}_{2,\dot{\alpha}} \rightarrow \frac{i}{\sqrt{2}}(\partial_\mu \psi \sigma^\mu)_{\dot{\alpha}}, & \psi_2^\alpha \rightarrow 0, & d_2 \rightarrow -\frac{1}{4}\Box A \end{cases} .$$

The result of this exercise is:

$$\hat{\phi}^\dagger \hat{\phi}\big|_{(\theta\theta)(\bar{\theta}\bar{\theta})} = -\frac{1}{4}A^\star \Box A - \frac{1}{4}A\,\Box A^\star + \frac{1}{2}\partial_\mu A^\star \partial^\mu A - \frac{i}{2}(\partial_\mu \psi \sigma^\mu \bar{\psi}) + \frac{i}{2}(\psi \sigma^\mu \partial_\mu \bar{\psi}) + F^\star F \,. \tag{152}$$

We get kinetic terms for the scalar field A and and for the Weyl spinor ψ. However, there is no kinetic term for the scalar F. We will see in the next section that this field F is auxiliary in nature.

4 Supersymmetric Field Theories

Finally, we can put all the building blocks together and construct supersymmetric field theories. We will first discuss interacting chiral and anti-chiral superfields, i.e. the Wess Zumino model [4], and then supersymmetric gauge theories.

4.1 Wess Zumino Model

The possibly simplest case of an interacting supersymmetric model consists of a chiral and anti-chiral superfield with kinetic term and a supersymmetric interaction potential, the so-called super potential. In order to make it not too simple, we readily consider a collection of N chiral and anti-chiral superfields $\hat{\phi}_i$ and $\hat{\phi}_i^\dagger$. Here's the Lagrangian (summation over doubly appearing indices is understood):

$$\mathcal{L}_{\rm WZ} = \hat{\phi}_i^\dagger \hat{\phi}_i \big|_{(\theta\theta)(\bar{\theta}\bar{\theta})} + \underbrace{\left[a_i \hat{\phi}_i + \frac{1}{2} m_{ij} \hat{\phi}_i \hat{\phi}_j + \frac{1}{3!} g_{ijk} \hat{\phi}_i \hat{\phi}_j \hat{\phi}_k \right]}_{=\text{superpotential } W(\hat{\phi}_i)} \Bigg|_{(\theta\theta)} + \left[W(\hat{\phi}_i) \right]^\dagger \Bigg|_{(\bar{\theta}\bar{\theta})} . \qquad (153)$$

Note that we will assume, without loss of generality, that m_{ij} and g_{ijk} are symmetric under any permutation of their indices. This is, because the products of superfields here do not depend on the order of the factors.

By construction, each term in the Lagrangian $\mathcal{L}_{\rm WZ}$ varies by a total derivative under supersymmetry transformations and leaves therefore the action

$$S = \int_V d^4x \, \mathcal{L}_{\rm WZ}(x) \qquad (154)$$

invariant: The variation of the action under a supersymmetry transformation δS can be written as a surface integral over the boundary ∂V of the space-time volume V, which vanishes in the limit of the volume V covering the whole of space-time, because all fields are required to vanish at spatial infinity and for $t \to \pm\infty$.

The Wess Zumino Lagrangian does not contain powers of superfields higher than three. This is to insure renormalisability of the model. Including powers of superfields higher than three would introduce non-renormalisable interactions, as we will see shortly.

> Determine the Lagrangian $\mathcal{L}_{\rm WZ}$ in terms of the component fields A_i, ψ_i, F_i and their complex conjugates with the help of the product formulas of the section above.

After partial integration of the kinetic terms involving the d'Alembert operator ($\Box$), we get the result:

$$\begin{aligned}\mathcal{L}_{\mathrm{WZ}} = \partial_\mu A_i^\star \partial^\mu A_i - \frac{i}{2}(\partial_\mu \psi_i \sigma^\mu \bar{\psi}_i) + \frac{i}{2}(\psi_i \sigma^\mu \partial_\mu \bar{\psi}_i) + F_i^\star F_i \\ + a_i F_i + m_{ij}(A_i F_j - \frac{1}{2}(\psi_i \psi_j)) + \frac{1}{2} g_{ijk}(A_i A_j F_k - A_i(\psi_j \psi_k)) \\ + a_i^\star F_i^\star + m_{ij}^\star (A_i^\star F_j^\star - \frac{1}{2}(\bar{\psi}_i \bar{\psi}_j)) + \frac{1}{2} g_{ijk}^\star (A_i^\star A_j^\star F_k^\star - A_i^\star(\bar{\psi}_j \bar{\psi}_k)) \,.\end{aligned} \tag{155}$$

As indicated before, there are no derivative terms involving the fields F_i and $F_i^\star$ present in the Lagrangian Hence, the field equations for those fields

$$\partial_\mu \frac{\partial \mathcal{L}_{\mathrm{WZ}}}{\partial(\partial_\mu F_l)} - \frac{\partial \mathcal{L}_{\mathrm{WZ}}}{\partial F_l} = 0 \,, \qquad \partial_\mu \frac{\partial \mathcal{L}_{\mathrm{WZ}}}{\partial(\partial_\mu F_l^\star)} - \frac{\partial \mathcal{L}_{\mathrm{WZ}}}{\partial F_l^\star} = 0 \,, \tag{156}$$

reduce to

$$\frac{\partial \mathcal{L}_{\mathrm{WZ}}}{\partial F_l} = 0 \,, \qquad \frac{\partial \mathcal{L}_{\mathrm{WZ}}}{\partial F_l^\star} = 0 \,. \tag{157}$$

Using the Lagrangian $\mathcal{L}_{\mathrm{WZ}}$ in Eq. (155), determine the field equations for the fields F_l and $F_l^\star$.

The result gives algebraic relations between the fields F_l and the $A_i^\star$, and between the $F_l^\star$ and the A_i:

$$F_l^\star = -a_l - m_{il} A_i - \frac{1}{2} g_{ijl} A_i A_j \,, \tag{158}$$

$$F_l = -a_l^\star - m_{il}^\star A_i^\star - \frac{1}{2} g_{ijl}^\star A_i^\star A_j^\star \,. \tag{159}$$

Now the auxiliary nature of the the fields F_l and $F_l^\star$ is apparent: We can re-insert the algebraic relations above in the Lagrangian of Eq. (155) and eliminate the fields F_l and $F_l^\star$ completely. This form of the Lagrangian is called the "on-shell" Lagrangian, because the auxiliary fields have been eliminated.

Starting from the Wess Zumino Lagrangian for one superfield without linear term in the superpotential and real parameters,

$$\mathcal{L}_{\mathrm{WZ}} = \hat{\phi}^\dagger \hat{\phi}\big|_{(\theta\theta)(\bar{\theta}\bar{\theta})} + \left[\frac{1}{2} m \hat{\phi}^2 + \frac{1}{3!} g \hat{\phi}^3\right]\Bigg|_{(\theta\theta)} + \left[\frac{1}{2} m (\hat{\phi}^\dagger)^2 + \frac{1}{3!} g (\hat{\phi}^\dagger)^3\right]\Bigg|_{(\bar{\theta}\bar{\theta})} \,,$$

determine the on-shell Lagrangian, i.e. eliminate the auxiliary fields F and $F^\star$ in the Lagrangian in component fields with the help of the field equations they obey.

The result of this exercise is:

$$\mathcal{L}_{\text{WZ,on-shell}} = \partial_\mu A^\star \partial^\mu A - m^2 A^\star A - \frac{i}{2}(\partial_\mu \psi_i \sigma^\mu \bar{\psi}_i) + \frac{i}{2}(\psi_i \sigma^\mu \partial_\mu \bar{\psi}_i) - \frac{1}{2}m(\psi\psi + \bar{\psi}\bar{\psi})$$
$$- \frac{1}{2}gA\psi\psi - \frac{1}{2}gA^\star\bar{\psi}\bar{\psi} - \frac{1}{2}mg(A^\star)^2 A - \frac{1}{2}mgA^\star A^2 - \frac{1}{4}g^2(A^\star)^2 A^2 \,. \qquad (160)$$

A few typical features of supersymmetric models are apparent here which we highlight here:

- There is a complex scalar A and a Weyl spinor field ψ in the particle spectrum with the same mass m .
- Upon eliminating the auxiliary fields, a scalar four-point interaction of A is generated.
- The Yukawa coupling between the A and ψ and the four-point interaction of A are determined by the same coupling constant g.
- The scalar three-point interaction of A is determined by the mass m and g.

This conspiracy of couplings and masses leads for instance to the cancellation of quadratically divergent loop contributions to self energies.

Note that the "on-shell" Lagrangian would not have looked much different, if we had kept a term linear in the superfield in the super potential:

$$W(\hat{\phi}) = a\hat{\phi} + \frac{1}{2}m\hat{\phi}^2 + \frac{1}{3!}g\hat{\phi}^3 \,.$$

With a shift of A by a constant, $A \to A + A_0$, which does not spoil the chirality and anti-chirality of $\hat{\phi}$ and $\hat{\phi}^\dagger$, one also arrives at the form of the Lagrangian in Eq. (160) but with m shifted to:

$$m \to M = \sqrt{m^2 - 2ga} \,.$$

4.2 Abelian Gauge Theory

4.2.1 Minimal Substitution, Matter Gauge Interaction

In ordinary free field theory of some complex scalar $\phi(x)$ or fermion $\psi(x)$, the corresponding kinetic and mass terms

$$\mathcal{L} = (\partial_\mu \phi)^\star \partial^\mu \phi - m^2 \phi^\star \phi \qquad \text{or} \qquad \mathcal{L} = \bar{\psi}\, i\gamma_\mu \partial^\mu \psi - m\bar{\psi}\psi$$

are invariant under *global* phase transformations:

$$\begin{aligned} \phi &\to \phi' = e^{i\alpha}\phi \,, \\ \phi^\star &\to \phi'^\star = e^{-i\alpha}\phi^\star \,, \end{aligned} \qquad \text{or} \qquad \begin{aligned} \psi &\to \psi' = e^{i\alpha}\psi \,, \\ \bar{\psi} &\to \bar{\psi}' = e^{-i\alpha}\bar{\psi} \,, \end{aligned}$$

with an arbitrary phase α which is constant in space and time.

Requiring *local* phase invariance, i.e. allowing the phase to depend on the space-time coordinate, $\alpha \to \alpha(x)$, the minimal substitution $\partial_\mu \to \partial_\mu - igv_\mu(x)$ in the corresponding kinetic term leads to Abelian gauge interactions of this scalar or fermion with a gauge field $v_\mu(x)$. Local phase invariance is then realised by the gauge field transforming locally with a shift proportional to the gradient of the phase function $\alpha(x)$:

$$v_\mu(x) \to v'_\mu(x) = v_\mu(x) + \frac{1}{g}\partial_\mu \alpha(x)\,.$$

The supersymmetric generalisation of the minimal substitution should also start from the kinetic term. For chiral superfields, which will be the building blocks for the "supersymmetrisation" of the SM (chiral) fermions and the Higgs field, we showed that it reads:

$$\hat{\phi}^\dagger \hat{\phi}\Big|_{(\theta\theta)(\bar{\theta}\bar{\theta})}$$

Obviously, this term is invariant under global phase transformations:

$$\hat{\phi} \to \hat{\phi}' = e^{i\alpha}\hat{\phi}\,, \qquad \hat{\phi}^\dagger \to \hat{\phi}'^\dagger = e^{-i\alpha}\hat{\phi}^\dagger\,.$$

Moreover, multiplying a chiral superfield by a constant does not spoil its property of being a chiral superfield. In fact, a scalar field ϕ_c, constant in space and time, can be considered a chiral and anti-chiral superfield, as it fulfils both $\bar{D}_{\dot{\alpha}}\phi_c = 0$ and $D_\alpha \phi_c = 0$. However, as soon as we allow for local phase transformations, it is clear that we need to extend the transformation in a supersymmetric way, because the product $e^{i\alpha(x)}\hat{\phi}(z)$ is not a chiral superfield anymore. We know already from Section 3.6 that any multiple product of chiral superfields is again a chiral superfield. Hence, the way to remedy the situation is to introduce a full chiral superfield $\hat{\Lambda}(z)$ (i.e. $\bar{D}_{\dot{\alpha}}\hat{\Lambda} = 0$) and its complex conjugate $\hat{\Lambda}^\dagger(z)$, which is anti-chiral ($D_\alpha \hat{\Lambda}^\dagger = 0$), as the transformation parameter, i.e. the concept of gauge transformations needs a supersymmetric generalisation, as already indicated in the section on the vector superfield above:

$$\hat{\phi} \to \hat{\phi}' = e^{i\hat{\Lambda}(z)}\hat{\phi}\,, \qquad \hat{\phi}^\dagger \to \hat{\phi}'^\dagger = \hat{\phi}^\dagger e^{-i\hat{\Lambda}^\dagger(z)}\,.$$

Note that the order of the factors in this equations (e.g. of $\hat{\phi}^\dagger e^{-i\hat{\Lambda}^\dagger(z)}$) is not relevant, because, as we saw, the product of two scalar superfields (and the special cases derived from it) commute, and, in the Abelian case, we do not assume any other non-commuting structure for $\hat{\phi}(z)$ and $\hat{\Lambda}(z)$. In the non-Abelian case, the order of terms *does* matter, and the way the transformation is written above holds also in this case.

For a pair of chiral and anti-chiral superfields $\hat{\phi}$ and $\hat{\phi}^\dagger$, which we will denote as "matter fields" (in a broad sense), it turns out that the product

$$\hat{\phi}^\dagger e^{2g\hat{V}}\hat{\phi}\Big|_{(\theta\theta)(\bar{\theta}\bar{\theta})} \tag{161}$$

with a vector superfield $\hat{V}$ is gauge symmetric as well as supersymmetric if the vector superfield transforms according to the rule:

$$V(z) \to V'(z) = V(z) - \frac{1}{2g}\left(i\hat{\Lambda}(z) - i\hat{\Lambda}^\dagger(z)\right) . \tag{162}$$

To illustrate this, we write out the gauge transformed term in Eq. (161) explicitly:

$$\hat{\phi}'^\dagger e^{2g\hat{V}'}\hat{\phi}' = \hat{\phi}^\dagger e^{-i\hat{\Lambda}^\dagger(z)} e^{2g\hat{V}'} e^{i\hat{\Lambda}(z)}\hat{\phi} = \hat{\phi}^\dagger e^{2g\hat{V}'+i\hat{\Lambda}(z)-i\hat{\Lambda}^\dagger(z)}\phi \overset{(162)}{=} \hat{\phi}^\dagger e^{2g\hat{V}}\phi \, .$$

As the products of all involved superfields with each other commute, we were allowed to use the normal power rule $e^A e^B = e^{A+B}$. This, of course, is different in the non-Abelian case.

We will assume now, that the Wess Zumino gauge has been chosen, i.e. according to Eq. (140) we have:

$$\hat{V}(z)|_{\mathrm{WZ}} = -(\theta\sigma^\mu\bar{\theta})v_\mu(x) - i(\theta\theta)\bar{\theta}\bar{\lambda}(x) + i(\bar{\theta}\bar{\theta})\theta\lambda(x) - \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})D(x) \, .$$

From this, we also know that all powers of $\hat{V}_{\mathrm{WZ}}$ higher than two vanish. Thus we get:

$$e^{2g\hat{V}_{\mathrm{WZ}}} = 1 + 2g\hat{V}_{\mathrm{WZ}} + 2g^2\hat{V}_{\mathrm{WZ}}^2 \, .$$

Note that after choosing the Wess Zumino gauge class, supersymmetry is not manifest anymore. However, we still are free to choose an arbitrary function $\alpha(x)$, which is the real part of the lowest component (i.e. the one without Grassmann parameters) of $\hat{\Lambda}(z)$. We can see this by translating Eq. (138) and Eq. (139) using $\hat{\phi}(z) \to -i\hat{\Lambda}(z)/(2g)$ and $\hat{\phi}^\dagger(z) \to i\hat{\Lambda}^\dagger(z)/(2g)$:

$$\begin{aligned}\frac{1}{g}\alpha(x) := -2\mathrm{Im}(A(x)) &= -\frac{1}{i}(\hat{\phi}(x,0,0) - \hat{\phi}^\dagger(x,0,0)) \\ &= \frac{i}{2g}(-i\Lambda(x,0,0) - i\Lambda^\dagger(x,0,0)) = \frac{1}{g}\mathrm{Re}(\Lambda(x,0,0)) \, .\end{aligned}$$

This freedom allows us to perform the usual gauge transformations of the component fields A, ψ and v_μ:

$$\begin{aligned} A(x) &\to A'(x) = e^{i\alpha(x)}A(x) \, , & \psi(x) &\to \psi'(x) = e^{i\alpha(x)}\psi(x) \, , \\ A^\star(x) &\to A'^\star(x) = e^{-i\alpha(x)}A^\star(x) \, , & \bar{\psi}(x) &\to \bar{\psi}'(x) = e^{-i\alpha(x)}\bar{\psi}(x) \, , \\ v_\mu(x) &\to v'_\mu(x) = v_\mu(x) + \frac{1}{g}\alpha(x) \, . \end{aligned}$$

Convince yourself that

$$\mathcal{L}_{\text{matter gauge int.}} = \hat{\phi}^\dagger e^{2g\hat{V}_{\text{WZ}}}\hat{\phi}\Big|_{(\theta\theta)(\bar{\theta}\bar{\theta})} = \left[\hat{\phi}^\dagger\hat{\phi} + 2g\hat{\phi}^\dagger\hat{V}_{\text{WZ}}\hat{\phi} + 2g^2\hat{\phi}^\dagger\hat{V}_{\text{WZ}}^2\hat{\phi}\right]\Big|_{(\theta\theta)(\bar{\theta}\bar{\theta})} \tag{163}$$

leads to a gauge invariant field theory in component fields by expressing $\mathcal{L}$ in component fields. We have already calculated the $(\theta\theta)(\bar{\theta}\bar{\theta})$ component of $\hat{\phi}^\dagger\hat{\phi}$ in component fields in Eq. (152). So, the real work to be done is to calculate the products

$$\hat{\phi}^\dagger\hat{V}_{\text{WZ}}\hat{\phi}\Big|_{(\theta\theta)(\bar{\theta}\bar{\theta})} \qquad \text{and} \qquad \hat{\phi}^\dagger\hat{V}_{\text{WZ}}^2\hat{\phi}\Big|_{(\theta\theta)(\bar{\theta}\bar{\theta})}$$

making heavy use of our general product formula Eq. (145). In order to simplify this task, let us note here the component entries of a standard scalar superfield for $\hat{\phi}, \hat{\phi}^\dagger$ and $\hat{V}_{\text{WZ}}$:

$$\hat{\phi}^\dagger : \left\{ \begin{array}{lllll} f \to A^\star, & \phi_\alpha \to 0, & & \bar{\chi}_{\dot{\alpha}} \to \sqrt{2}\bar{\psi}_{\dot{\alpha}}, \\ m \to 0, & n \to F^\star, & & v_\mu \to i\partial_\mu A^\star, \\ \bar{\lambda}^{\dot{\alpha}} \to 0, & \psi_\alpha \to -\frac{i}{\sqrt{2}}{}_\alpha(\sigma^\mu\partial_\mu\bar{\psi}), & & d \to -\frac{1}{4}\Box A^\star \end{array} \right\},$$

$$\hat{\phi} : \left\{ \begin{array}{lll} f \to A, & \phi_\alpha \to \sqrt{2}\psi_\alpha, & \bar{\chi}_{\dot{\alpha}} \to 0, \\ m \to F, & n \to 0, & v_\mu \to -i\partial_\mu A, \\ \bar{\lambda}_{\dot{\alpha}} \to \frac{i}{\sqrt{2}}(\partial_\mu\psi\sigma^\mu)_{\dot{\alpha}}, & \psi^\alpha \to 0, & d \to -\frac{1}{4}\Box A \end{array} \right\},$$

$$\hat{V}_{\text{WZ}} : \left\{ \begin{array}{lll} f \to 0, & \phi_\alpha \to 0, & \bar{\chi}_{\dot{\alpha}} \to 0, \\ m \to 0, & n \to 0, & v_\mu \to -v_\mu, \\ \bar{\lambda}_{\dot{\alpha}} \to -i\bar{\lambda}_{\dot{\alpha}}, & \psi^\alpha \to i\lambda_\alpha, & d \to -\frac{1}{2}D \end{array} \right\}.$$

Some intermediate results of this exercise are:

$$\hat{V}_{\text{WZ}}^2 = \frac{1}{2}v_\mu v^\mu(\theta\theta)(\bar{\theta}\bar{\theta}),$$

$$\hat{\phi}^\dagger\hat{V}_{\text{WZ}}\hat{\phi}\Big|_{(\theta\theta)(\bar{\theta}\bar{\theta})} = -\frac{1}{2}A^\star DA + \frac{i}{2}(A^\star\partial^\mu A - A\partial^\mu A^\star)v_\mu - \frac{i}{\sqrt{2}}(A^\star\psi\lambda - A\bar{\psi}\bar{\lambda}) + \frac{1}{2}(\bar{\psi}\bar{\sigma}^\mu\psi)v_\mu .$$

Putting everything together, we get finally:

$$
\begin{aligned}
\mathcal{L}_{\text{matter gauge int.}} &= \big[\left(\partial_\mu - igv_\mu(x)\right) A(x) \big]^\star \big[\left(\partial^\mu - igv_\mu(x)\right) A(x) \big] \\
&\quad + i\bar{\psi}(x)\bar{\sigma}^\mu(\partial^\mu - igv_\mu(x))\psi(x) \\
&\quad - \sqrt{2}g \left(A^\star(x)\psi(x)\lambda(x) - A(x)\bar{\psi}(x)\bar{\lambda}(x) \right) \\
&\quad - gA^\star(x)D(x)A(x) + F^\star(x)F(x) \,.
\end{aligned} \tag{164}
$$

While a full model needs the supersymmetric version of the gauge field dynamics too, which we will develop below, a few remarks seem fit at this point:

- Gauge invariance is apparent: Each partial derivative is accompanied by a gauge field term to form a gauge covariant derivative, as required by gauge symmetry.
- The scalar field A and its fermionic partner ψ of the chiral multiplet $\hat{\phi}$ are both gauge fields with the same U(1) charge.
- There is an interaction which connects the scalar A, the fermion ψ and the fermion λ which is a fermionic member of the vector superfield.
- Apart from F, which we know to be auxiliary, the fields D and λ do not have any dynamics so far. As it will turn out shortly, D is also an auxiliary field, while λ is a dynamical degree of freedom of the vector superfield, the fermionic superpartner of the vector field (the "gaugino"), which receives its dynamics from the kinetic term of the gauge field.

4.2.2 Kinetic Term of the Gauge Field

This section is most likely beyond the scope of the exercise course given the time constraints. For completeness, it is included anyway and even some exercises are put in. The construction of the supersymmetric kinetic term of the gauge field proceeds in an even less transparent way than the construction in the previous section. The following remarks try to motivate it.

Already when we discussed the vector superfield, we realised that the fields $\lambda^\alpha(x)$ and $D(x)$ do not change under a supersymmetric gauge transformation. A superfield $\hat{W}_\alpha$ and its complex conjugate $\bar{\hat{W}}_{\dot{\alpha}}$, constructed from the vector superfield $\hat{V}$, which has $\lambda^\alpha(x)$ and $\bar{\lambda}^{\dot{\alpha}}(x)$, respectively, as its lowest component can be obtained via multiple applications of the SUSY covariant derivatives in the following form:

$$
\hat{W}_\alpha = -\frac{1}{4}(\bar{D}\bar{D})D_\alpha\hat{V} \qquad \text{and} \qquad \bar{\hat{W}}_{\dot{\alpha}} = -\frac{1}{4}(DD)\bar{D}_{\dot{\alpha}}\hat{V} \,. \tag{165}
$$

Use the algebra of the D's and $\bar{D}$'s in Eq. (110) to show that the superfields $\hat{W}_\alpha$ and $\hat{\bar{W}}_{\dot{\alpha}}$ are chiral and anti-chiral superfields, respectively, i.e. show:

$$\bar{D}_{\dot{\alpha}}\hat{W}_\alpha = 0 \qquad \text{and} \qquad D_\alpha \hat{\bar{W}}_{\dot{\alpha}} = 0\,.$$

The solution is very simple. We use the fact that products of squares of anti-commuting factors vanish. Writing out the differential operators we get:

$$\bar{D}_{\dot{\alpha}}\hat{W}_\alpha = -\frac{1}{4}\bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}\bar{D}_{\dot{\gamma}}\bar{\epsilon}^{\dot{\gamma}\dot{\beta}}D_\alpha\hat{V}\,.$$

From the anti-commutation property of the $\bar{D}$'s in Eq. (110) we know that $(\bar{D}_{\dot{1}})^2 = (\bar{D}_{\dot{2}})^2 = 0$ and, as the index space of $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$ is only $\{\dot{1}, \dot{2}\}$, there must always be two equal factors in $\bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}\bar{D}_{\dot{\gamma}}$. Hence,

$$\bar{D}_{\dot{\alpha}}\hat{W}_\alpha = 0\,, \qquad \text{and likewise} \qquad D_\alpha \hat{\bar{W}}_{\dot{\alpha}} = 0\,.$$

Knowing that $\hat{W}_\alpha$ and $\hat{\bar{W}}_{\dot{\alpha}}$ are chiral and anti-chiral superfields, a supersymmetric Lagrangian would be

$$\mathcal{L}_{\text{gauge kin.}} \propto \hat{W}^\alpha\hat{W}_\alpha\big|_{(\theta\theta)} + \hat{\bar{W}}_{\dot{\alpha}}\hat{\bar{W}}^{\dot{\alpha}}\big|_{(\bar{\theta}\bar{\theta})}\,. \tag{166}$$

However, before we jump to conclusions, we first have to see if this construction is also invariant under supersymmetric gauge transformations.

Use the algebra of the D's and $\bar{D}$'s in Eq. (109) and Eq. (110) to show that the superfields $\hat{W}_\alpha$ and $\hat{\bar{W}}_{\dot{\alpha}}$ are invariant under the supersymmetric gauge transformation

$$\hat{V} \to \hat{V}' = \hat{V} - \frac{1}{2g}\left(i\hat{\Lambda} - i\hat{\Lambda}^\dagger\right)\,.$$

Note that $\hat{\Lambda}(z)$ and $\hat{\Lambda}^\dagger(z)$ are chiral and anti-chiral superfields, i.e.

$$\bar{D}_{\dot{\alpha}}\hat{\Lambda} = 0\,, \qquad D_\alpha\hat{\Lambda}^\dagger = 0\,. \tag{167}$$

We show here the solution for $\hat{W}_\alpha$ (The reasoning is similar for $\hat{\bar{W}}_{\dot{\alpha}}$.):

$$\begin{aligned}\hat{W}_\alpha \to & -\frac{1}{4}(\bar{D}\bar{D})D_\alpha\left(\hat{V} - \frac{i}{2g}\hat{\Lambda} + \frac{i}{2g}\hat{\Lambda}^\dagger\right), \\ &= \hat{W}_\alpha + \frac{i}{8g}(\bar{D}\bar{D})D_\alpha\hat{\Lambda}, \quad (D_\alpha\hat{\Lambda}^\dagger = 0) \\ &= \hat{W}_\alpha + \frac{i}{8g}\bar{\epsilon}^{\dot{\beta}\dot{\alpha}}\bar{D}_{\dot{\alpha}}\{\bar{D}_{\dot{\beta}}, D_\alpha\}\hat{\Lambda}, \quad (\bar{D}_{\dot{\alpha}}\hat{\Lambda} = 0) \\ &= \hat{W}_\alpha + \frac{i}{8g}\bar{\epsilon}^{\dot{\beta}\dot{\alpha}}2i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{D}_{\dot{\alpha}}\hat{\Lambda}, \quad (\text{Eq. (109)}) \\ &= \hat{W}_\alpha\,. \quad (\bar{D}_{\dot{\alpha}}\hat{\Lambda} = 0)\end{aligned}$$

Up to now, we showed that $\hat{W}_\alpha$ and $\hat{\bar{W}}_{\dot{\alpha}}$ are chiral and anti-chiral superfields, respectively, and both fields are invariant under supersymmetric gauge transformations. Hence, we know now that Eq. (166) gives a viable Lagrangian. In order to determine the correct normalisation of the terms in the Lagrangian, we determine now the product $W^\alpha W_\alpha|_{(\theta\theta)}$ in Wess Zumino gauge. This is a lengthy calculation. We just quote here a few milestones on the way to the final result:

$$\begin{aligned}\bar{D}\bar{D} &= -\bar{\partial}_{\dot{\alpha}}\bar{\partial}^{\dot{\alpha}} + 2i(\theta\sigma^\mu)_{\dot{\alpha}}\bar{\partial}^{\dot{\alpha}}\partial_\mu + (\theta\theta)\Box\,, \\ \hat{W}_\alpha &= i\lambda_\alpha + \theta^\beta\left[\epsilon_{\alpha\beta}D + \frac{1}{2}(\sigma^{\mu\nu})_\alpha{}^\gamma\epsilon_{\beta\gamma}(\partial_\mu v_\nu - \partial_\nu v_\mu)\right] + (\theta\theta)\,{}_\alpha(\sigma^\mu\partial_\mu\bar{\lambda}) \\ &\quad + (\theta\sigma^\mu\bar{\theta})\partial_\mu\lambda_\alpha + (\theta\theta)\bar{\theta}^{\dot{\beta}}\left[-\frac{i}{2}(\sigma^\mu)_{\alpha\dot{\beta}}D - (\sigma^\rho\bar{\sigma}^\nu\sigma^\mu)\partial_\rho\partial_\mu v_\nu + \frac{1}{4}(\sigma^\mu)_{\alpha\dot{\beta}}\Box v_\mu\right] \\ &\quad + (\theta\theta)(\bar{\theta}\bar{\theta})\left[-\frac{i}{4}\Box\lambda_\alpha\right].\end{aligned}$$

In order to arrive at the final result also the following relation is needed:

$$(\sigma^{\mu\nu}\sigma^{\kappa\lambda})_\alpha{}^\alpha = 2g^{\mu\kappa}g^{\nu\lambda} - 2g^{\mu\lambda}g^{\nu\kappa} + 2i\epsilon^{\mu\nu\kappa\lambda}\,.$$

So, we get (with $v_{\mu\nu} := \partial_\mu v_\nu - \partial_\nu v_\mu$):

$$\hat{W}^\alpha\hat{W}_\alpha\big|_{(\theta\theta)} = -\frac{1}{2}v_{\mu\nu}v^{\mu\nu} + 2i(\lambda\sigma^\mu\partial_\mu\bar{\lambda}) + D^2 - \frac{i}{4}\epsilon^{\mu\nu\kappa\lambda}v_{\mu\nu}v_{\kappa\lambda}\,, \tag{168}$$

and by complex conjugation

$$\hat{\bar{W}}_{\dot{\alpha}}\hat{\bar{W}}^{\dot{\alpha}}\big|_{(\bar{\theta}\bar{\theta})} = -\frac{1}{2}v_{\mu\nu}v^{\mu\nu} - 2i(\partial_\mu\lambda\sigma^\mu\bar{\lambda}) + D^2 + \frac{i}{4}\epsilon^{\mu\nu\kappa\lambda}v_{\mu\nu}v_{\kappa\lambda}\,. \tag{169}$$

Finally, we can write down the supersymmetric kinetic term of the gauge superfield in Wess Zumino gauge:

$$\mathcal{L}_{\text{gauge kin.}} = \frac{1}{4}\left(\hat{W}^\alpha\hat{W}_\alpha\big|_{(\theta\theta)} + \hat{\bar{W}}_{\dot{\alpha}}\hat{\bar{W}}^{\dot{\alpha}}\big|_{(\bar{\theta}\bar{\theta})}\right), \tag{170}$$

$$= -\frac{1}{4}v_{\mu\nu}v^{\mu\nu} + \frac{i}{2}(\lambda\sigma^\mu\partial_\mu\bar{\lambda}) - \frac{i}{2}(\partial_\mu\lambda\sigma^\mu\bar{\lambda}) + \frac{1}{2}D^2\,. \tag{171}$$

Some remarks:

- The gauge field v_μ appears in the usual antisymmetric combination $v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$, which is invariant under the remaining gauge symmetry within the Wess Zumino gauge class.
- A kinetic term for the fermion field λ and its complex conjugate appears. This field is called the superpartner to the gauge field v_μ, the "gaugino."
- The field D is indeed an auxiliary field. It has no dynamics, i.e. there are no derivatives acting on D.

4.2.3 The Full Lagrangian

We would like to put together the most general model we can construct so far. We include the supersymmetric Lagrangians for the gauge interaction of "matter fields" described by chiral superfields $\hat{\Phi}_i$ with U(1) charges Q_i (some of which may be zero) and the gauge kinetic terms. We also include a superpotential of the chiral superfields, the couplings of which we assume to observe the gauge symmetry. We get for the manifest supersymmetric and gauge invariant Lagrangian form:

$$\mathcal{L} = \hat{\phi}_i^\dagger e^{2gQ_i\hat{V}}\hat{\phi}_i\Big|_{(\theta\theta)(\bar{\theta}\bar{\theta})} + W(\hat{\phi}_i)\big|_{(\theta\theta)} + W(\hat{\phi}_i^\dagger)\big|_{(\bar{\theta}\bar{\theta})} + \frac{1}{4}\left(\hat{W}^\alpha\hat{W}_\alpha\big|_{(\theta\theta)} + \hat{\bar{W}}_{\dot{\alpha}}\hat{\bar{W}}^{\dot{\alpha}}\big|_{(\bar{\theta}\bar{\theta})}\right), \quad (172)$$

with

$$\hat{W}_\alpha = -\frac{1}{4}(\bar{D}\bar{D})D_\alpha\hat{V} \qquad \hat{\bar{W}}_{\dot{\alpha}} = -\frac{1}{4}(DD)\bar{D}_{\dot{\alpha}}\hat{V}\,, \quad (173)$$

and the superpotential (not taking into account terms linear in superfields)

$$W(\hat{\phi}_i) = \frac{1}{2}m_{ij}\hat{\phi}_i\hat{\phi}_j + \frac{1}{3!}g_{ijk}\hat{\phi}_i\hat{\phi}_j\hat{\phi}_k\,. \quad (174)$$

Note that the parameters of the superpotential have to observe some constraints, lest they spoil the gauge symmetry:

- $m_{ij} = 0$ if $Q_i + Q_j \neq 0$
- $g_{ijk} = 0$ if $Q_i + Q_j + Q_k \neq 0$

In Wess Zumino gauge, we get for the "off-shell" Lagrangian in component fields (sums over i, j, k implied):

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}v_{\mu\nu}v^{\mu\nu} + \frac{i}{2}(\lambda\sigma^{\mu}\partial_{\mu}\bar{\lambda}) - \frac{i}{2}(\partial_{\mu}\lambda\sigma^{\mu}\bar{\lambda}) \\ &+ \left[(\partial_{\mu} - igQ_i v_{\mu}) A_i \right]^{\star} \left[(\partial^{\mu} - igQ_i v_{\mu}) A_i \right] + i\bar{\psi}_i\bar{\sigma}^{\mu}(\partial^{\mu} - igQ_i v_{\mu})\psi_i \\ &- \frac{1}{2}m_{ij}(\psi_i\psi_j) - \frac{1}{2}m^{\star}_{ij}(\bar{\psi}_i\bar{\psi}_j) - \frac{1}{2}g_{ijk}A_i(\psi_j\psi_k) - \frac{1}{2}g^{\star}_{ijk}A^{\star}_i(\bar{\psi}_j\bar{\psi}_k) \\ &- \sqrt{2}gQ_i \left(A^{\star}_i\psi_i\lambda - A_i\bar{\psi}_i\bar{\lambda} \right) \\ &+ \frac{1}{2}D^2 - gQ_iA^{\star}_iDA_i \\ &+ F^{\star}_iF_i + (m_{ij}A_j + \frac{1}{2}g_{ijk}A_kA_j)F_i + (m^{\star}_{ij}A^{\star}_j + \frac{1}{2}g^{\star}_{ijk}A^{\star}_kA^{\star}_j)F^{\star}_i \,.\end{aligned} \tag{175}$$

The field equation for D can now be given:

$$D = g\sum_i Q_iA^{\star}_iA_i \,. \tag{176}$$

Thus, the D-dependent part of $\mathcal{L}$ reads "on-shell":

$$\mathcal{L}_D := \frac{1}{2}D^2 - g\sum_i Q_iA^{\star}_iDA_i = -\frac{1}{2}g^2(\sum_i Q_iA^{\star}_iA_i)^2 \,. \tag{177}$$

Using the field equations for $F^{\star}_i$ and F_i from Eq. (158) and Eq. (159) with $a_i = 0$:

$$\begin{aligned}F^{\star}_i &= -m_{ij}A_j - \frac{1}{2}g_{ijk}A_kA_j \,, \\ F_i &= -m^{\star}_{ij}A^{\star}_j - \frac{1}{2}g^{\star}_{ijk}A^{\star}_kA^{\star}_j \,,\end{aligned}$$

we can write the F_i,$F^{\star}_i$ dependent part of $\mathcal{L}$ also "on-shell":

$$\begin{aligned}\mathcal{L}_F &:= F^{\star}_iF_i + (m_{ij}A_j + \frac{1}{2}g_{ijk}A_kA_j)F_i + (m^{\star}_{ij}A^{\star}_j + \frac{1}{2}g^{\star}_{ijk}A^{\star}_kA^{\star}_j)F^{\star}_i \\ &= -\left(m^{\star}_{ij}A^{\star}_j + \frac{1}{2}g^{\star}_{ijk}A^{\star}_kA^{\star}_j\right)\left(m_{il}A_l + \frac{1}{2}g_{ilm}A_mA_l\right) \\ &= -m^{\star}_{ji}m_{il}A^{\star}_jA_l - \frac{1}{2}m^{\star}_{ij}g_{ilm}A^{\star}_jA_mA_l - \frac{1}{2}m_{il}g^{\star}_{ijk}A^{\star}_kA^{\star}_jA_l - \frac{1}{4}g^{\star}_{ijk}g_{ilm}A^{\star}_kA^{\star}_jA_mA_l\end{aligned} \tag{178}$$

Thus, we can eliminate all auxiliary fields in the full Lagrangian and write down the "on-shell" version:

$$
\begin{aligned}
\mathcal{L} = & -\frac{1}{4}v_{\mu\nu}v^{\mu\nu} + \frac{i}{2}(\lambda\sigma^{\mu}\partial_{\mu}\bar{\lambda}) - \frac{i}{2}(\partial_{\mu}\lambda\sigma^{\mu}\bar{\lambda}) \\
& + \left[(\partial_{\mu} - igQ_i v_{\mu})\, A_i\right]^{\star}\left[(\partial^{\mu} - igQ_i v_{\mu})\, A_i\right] + i\bar{\psi}_i\bar{\sigma}^{\mu}(\partial^{\mu} - igQ_i v_{\mu})\psi_i \\
& - \frac{1}{2}m_{ij}(\psi_i\psi_j) - \frac{1}{2}m^{\star}_{ij}(\bar{\psi}_i\bar{\psi}_j) - m^{\star}_{ji}m_{il}A^{\star}_j A_l \\
& - \frac{1}{2}g_{ijk}A_i(\psi_j\psi_k) - \frac{1}{2}g^{\star}_{ijk}A^{\star}_i(\bar{\psi}_j\bar{\psi}_k) - \frac{1}{2}m_{il}g^{\star}_{ijk}A^{\star}_k A^{\star}_j A_l - \frac{1}{2}m^{\star}_{ij}g_{ilm}A^{\star}_j A_m A_l \\
& - \sqrt{2}gQ_i\left(A^{\star}_i\psi_i\lambda - A_i\bar{\psi}_i\bar{\lambda}\right) \\
& - \frac{1}{4}g^{\star}_{ijk}g_{ilm}A^{\star}_k A^{\star}_j A_m A_l - \frac{1}{2}g^2\big(\sum_i Q_i A^{\star}_i A_i\big)^2 .
\end{aligned}
\tag{179}
$$

Additional remarks:

- The physical degrees of freedom of the chiral superfields are complex scalars A_i and corresponding Weyl fermions ψ_i with the same U(1) charge. They have masses, determined by the same mass matrix m_{ij} and. In fact, a redefinition of the superfields by a constant unitary matrix $\hat{\phi}'_i = U_{ij}\hat{\phi}_i$ allows to diagonalise the mass terms of the A_i's and the ψ_i's simultaneously. Hence, to each scalar described by A'_i corresponds a superpartner fermion ψ'_i of same mass and charge.

- There are Yukawa interactions of the scalars and fermions of the chiral superfields generated from the cubic part of the superpotential and corresponding three-scalar interactions with couplings proportional to the Yukawa couplings

- There are four-scalar interactions generated partly from the superpotential (from "F terms"), governed by the Yukawa couplings, and partly from the gauge interaction Lagrangian (from "D terms"), governed by gauge interactions. This is one of the typical features of supersymmetric field theories: There is no freedom to chose the four scalar interactions, they are all determined by Yukawa and gauge couplings. This leads to the often quoted cancellation of quadratic divergences in supersymmetric theories and, for instance, it also provides the reason why the lightest Higgs boson in the MSSM cannot be arbitrarily heavy.

———————————(Here ends the current write-up)———————————

The story goes on with the following topics:

Non-Abelian Supersymmetric Gauge Theory

SUSY Breaking

Spontaneous Breaking

Soft Breaking

The MSSM

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